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Bayesian Data Sketching for Spatial Regression Models

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SUMMARY

We introduce Bayesian data sketching for spatial regression models to obviate computational challenges presented by large numbers of spatial locations. To address the challenges of analysing very large spatial data, we compress spatially oriented data by a random linear transformation to achieve dimension reduction and conduct inference on the compressed data. Our approach distinguishes itself from several existing methods for analysing large spatial data in that it requires neither the development of new models or algorithms nor any specialised computational hardware while delivering fully model-based Bayesian inference. Well-established methods and algorithms for spatial regression models can be applied to the compressed data. We establish posterior contraction rates for estimating the spatially varying coefficients and predicting the outcome at new locations under the randomly compressed data model. We use simulation experiments and conduct a spatial analysis of remote sensed vegetation data to empirically illustrate the inferential and computational efficiency of our approach.

Some key words: Bayesian inference; B-splines; Data sketching; Predictive Process; Posterior contraction; Random compression matrix; Varying coefficient models.

1. INTRODUCTION

We develop an inferential framework for spatial data analysis using Bayesian data sketching to achieve scalable inference for massive spatial data sets. "Data sketching" (Vempala, 2005; Halko et al., 2011; Mahoney, 2011; Woodruff, 2014; Guhaniyogi & Dunson, 2015, 2016) is a method of compression that is being increasingly employed for analysing massive amounts of data. The entire data set is compressed before being analysed for computational efficiency. Data sketching proceeds by transforming the original data through a random linear transformation to produce a much smaller number of data samples and we conduct the analysis on the compressed

data thereby achieving dimension reduction. Furthermore, the original data is neither accessed nor exactly recoverable from the compressed data, which preserves data confidentiality.

While such developments have primarily focused on ordinary linear regression and penalised linear regression (Zhang et al., 2013; Chen et al., 2015; Dobriban & Liu, 2018; Drineas et al., 2011; Ahfock et al., 2017; Huang, 2018), our innovation lies in developing such methods for spatial regression models. The primary challenge distinguishing the current manuscript from existing data sketching methods is our pursuit of inference for the underlying spatial effects in the context of spatially-varying regression models. While bearing some similarities, our current contribution differs from compressed sensing (Donoho, 2006; Ji et al., 2008; Candes & Tao, 2006; Eldar & Kutyniok, 2012; Yuan et al., 2014) in the inferential objectives. Specifically, compressed sensing solves an inverse problem by "nearly" recovering a sparse vector of responses from a smaller set of random linear transformations. In contrast, our spatially referenced response vector is not necessarily sparse. Also, we do not seek to (approximately) so our method is applicable to situations where preserving confidentiality of the response (and predictors) is important.

We consider a spatially-varying regression model with response $y(s) \in \mathcal{Y} \subseteq \mathbb{R}$ and P predictors $x_1(s), ..., x_P(s) \in \mathcal{X} \subseteq \mathbb{R}$, $s \in \mathcal{D} \subseteq \mathbb{R}^2$ related according to the model

$$y(s) = \sum_{j=1}^{P} x_j(s)\beta_j + \sum_{j=1}^{\tilde{P}} \tilde{x}_j(s) w_j(s) + \epsilon(s) = x(s)^{\mathrm{T}}\beta + \tilde{x}(s)^{\mathrm{T}} w(s) + \epsilon(s) \ , \qquad (1)$$

where $\beta = (\beta_1, \beta_2, \ldots, \beta_P)^{\mathrm{T}}$ is a $P \times 1$ vector of spatially static coefficients, $\tilde{x}(s) = (\tilde{x}_1(s), \tilde{x}_2(s), \ldots, \tilde{x}_{\tilde{P}}(s))^{\mathrm{T}}$ is a $\tilde{P} \times 1$ vector comprising a subset of predictors from $x(s)$ (so $\tilde{P} \leq P$), $w(s) = (w_1(s), w_2(s), \ldots, w_{\tilde{P}}(s))^{\mathrm{T}}$ is the $\tilde{P} \times 1$ vector of spatially varying regression slopes, and $\epsilon(s) \overset{iid}{\sim} N(0, \sigma^2)$ captures measurement error variation at location s. Such spatially-varying regression coefficient models are effective tools for estimating the spatially varying impact of predictors on the response over space (see, e.g., Gelfand et al., 2003; Wheeler & Calder, 2007; Finley et al., 2011; Guhaniyogi et al., 2013; Kim & Wang, 2021, and references therein). Customary geostatistical regression models with only a spatially-varying intercept emerge if the first column of $x(s)$ is the intercept and $\tilde{P} = 1$ with $\tilde{x}_1(s) = 1$. Spatially-varying coefficient models also offer a process-based alternative to widely used geographically weighted regression (see, e.g., Brunsdon et al., 1996) for modelling nonstationary behaviour in the mean. Finley (2011) offers a comparative analysis and highlights the richness of (1) in ecological applications.

Bayesian inference for (1) is computationally expensive for large spatial data sets, as are commonplace today, due to the presence of the high-dimensional spatial covariance matrix introduced by $w(s)$ in (1). High-dimensional spatial modelling has been attracting significant interest and the burgeoning literature on diverse aspects of scalable methods is too vast to be comprehensively reviewed here (see, e.g., Banerjee, 2017; Heaton et al., 2019, for reviews). Briefly, model-based dimension reduction in spatial models have proceeded from low-rank or fixed rank representations (e.g., Cressie & Johannesson, 2008; Banerjee et al., 2008; Wikle, 2010), multi-resolution approaches (e.g., Nychka et al., 2015; Katzfuss, 2017; Guhaniyogi & Sansó, 2018), sparsity-inducing processes (e.g., Vecchia, 1988; Datta et al., 2016; Katzfuss & Guinness, 2021; Peruzzi et al., 2020) and divide-and-conquer approaches such as meta-kriging (Guhaniyogi & Banerjee, 2018; Guhaniyogi et al., 2020b). While most of the aforementioned methods entail new classes of models and approximations, or very specialised high-performance computing architectures, Bayesian data sketching has the advantage that customary exploratory data analysis tools, well-established methods and well-tested available algorithms for implementing (1) can be applied to the sketched data set without recourse to new algorithmic or software development.

We pursue fully model-based Bayesian data sketching, where inference proceeds from a hierarchical model (Cressie & Wikle, 2015; Banerjee et al., 2014). The hierarchical approach to spatial data analysis is widely employed for inferring on model parameters that may be weakly identified from the likelihood alone and, more relevantly for substantive inference, for estimating the latent spatial process over the domain of interest. For analytic tractability we model the varying coefficients using basis expansions (Wikle, 2010; Wang et al., 2008; Wang & Xia, 2009; Bai et al., 2019) rather than Gaussian processes. We exploit and adapt some recent developments in theory of random matrices to relate the inference from the compressed data with the full scale spatial model. We establish consistency of the posterior distributions of the spatially varying coefficients and analyse the predictive efficiency of our models based upon the compressed data. Posterior contraction of varying coefficient (VC) models have been investigated by a few recent articles. For example, Guhaniyogi et al. (2020a) derive minimax-optimal posterior contraction rates for Bayesian VC models under GP priors when the number of predictors P is fixed. Deshpande et al. (2020) also derived near-optimal posterior contraction rates under BART priors, and Bai et al. (2019) showed asymptotically optimal rate of estimation for varying coefficients with a variable selection prior on varying coefficients. We address these questions in the context of data compression, which has largely remained unexplored.

2. Bayesian Compressed Spatially Varying Coefficient Models

We model each spatially varying coefficient $w_j(s)$ in (1) as

$$w_j(s) = \sum_{h=1}^{H} B_{jh}(s)\gamma_{jh} \ , \quad j = 1, ..., \tilde{P} \ , \tag{2}$$

where each $B_{jh}(s)$ is a basis function evaluated at location s for $h = 1, ..., H$, and γ_{jh}'s are the corresponding basis coefficients. The distribution of these γ_{jh}'s yields a multivariate process with $\text{cov}(w_i(s), w_j(s')) = B_i(s)^{\text{T}}\text{cov}(\gamma_i, \gamma_j)B_j(s)$, where $B_i(s)$ and γ_i are $H \times 1$ with elements $B_{ih}(s)$ and γ_{ih}, respectively, for $h = 1, \ldots, H$.

Appropriate choices for basis functions can produce appropriate classes of multivariate spatial processes. A number of choices are available. For example, Biller & Fahrmeir (2001) and Huang et al. (2015) use splines to model the $B_{jh}(s)$'s and place Gaussian priors on the basis coefficients γ_{jh}. Li et al. (2015) propose a scale-mixture of multivariate normal distributions to shrink groups of basis coefficients towards zero. More recently, Bai et al. (2019) proposed using B-spline basis functions and multivariate spike-and-slab discrete mixture prior distributions on basis coefficients to aid functional variable selection. Other popular choices for basis functions include the wavelet basis (Vidakovic, 2009; Cressie & Wikle, 2015), radial basis (Bliznyuk et al., 2008) and locally bi-square (Cressie & Johannesson, 2008) or elliptical basis functions (Lemos & Sansó, 2009). Alternatively, a basis representation of $w_j(s)$ can be constructed by envisioning $w_j(s)$ as the projection of a Gaussian process $w_j(s)$ onto a set of reference locations, or "knots", which yields predictive processes and other variants (Banerjee et al., 2008; Guhaniyogi et al., 2013). More generally, each $w_j(s)$ can also be modelled using multi-resolution analogues to the aforesaid models to carefully capture global variations at the lower resolution and local variations at the higher resolutions (Katzfuss, 2017; Guhaniyogi & Sansó, 2018).

Let $\{y(s_i), x(s_i)\}$ be observations at N spatial locations $\mathcal{S} = \{s_1, s_2, \ldots, s_N\}$. Using (2) in (1) yields the Gaussian linear mixed model

$$y = X\beta + \tilde{X}B\gamma + \epsilon \ , \quad \epsilon \sim N(0, \sigma^2 I_N) \ . \tag{3}$$

where $y = (y(s_1), y(s_2), \ldots, y(s_N))^{\mathrm{T}}$ and $\epsilon = (\epsilon(s_1), \epsilon(s_2), \ldots, \epsilon(s_N))^{\mathrm{T}}$ are $N \times 1$ vectors of responses and errors, respectively, X is $N \times P$ with n-th row $x(s_n)^{\mathrm{T}}$, $\tilde{X}$ is the $N \times N\tilde{P}$ block-diagonal matrix with (n,n)-th block $\tilde{x}(s_n)^{\mathrm{T}}$, $B = (B(s_1)^{\mathrm{T}}, \ldots, B(s_N)^{\mathrm{T}})^{\mathrm{T}}$ is $N\tilde{P} \times H\tilde{P}$ with $B(s_n)$ a block-diagonal $\tilde{P} \times H\tilde{P}$ matrix whose j-th diagonal block is $(B_{j1}(s_n), \ldots, B_{jH}(s_n))$. The coefficient $\gamma = (\gamma_1^{\mathrm{T}}, ..., \gamma_{\tilde{P}}^{\mathrm{T}})^{\mathrm{T}}$ is $H\tilde{P} \times 1$ with each $\gamma_j = (\gamma_{j1}, \ldots, \gamma_{jH})^{\mathrm{T}}$ being $H \times 1$. Bayesian methods for estimating (3) typically employ a multivariate normal prior (Biller & Fahrmeir, 2001; Huang et al., 2015) or its scale-mixture (discrete as well as continuous) variants (Li et al., 2015; Bai et al., 2019) on γ.

Working with (3) will be expensive for large N. Instead, we consider data compression or sketching using a random linear mapping to reduce the size of the dataset from N to M observations. For this, we use M one-dimensional linear mappings of the data encoded by an $M \times N$ compression matrix Φ with $M << N$. This compression matrix is applied to y, X and $\tilde{X}$ to construct the $M \times 1$ compressed response vector $y_\Phi = \Phi y$ and the matrices $X_\Phi = \Phi X$ and $\tilde{X}_\Phi = \Phi\tilde{X}$. We will return to the specification of Φ, which, of course, will be crucial for relating the inference from the compressed data with the full model. For now assuming that we have fixed Φ, we construct a Bayesian hierarchical model for the compressed data

$$p(\psi, \beta, \gamma, \sigma^2 \,|\, y_\Phi, \Phi) \propto p(\psi, \sigma^2, \beta, \gamma) \times N(y_\Phi \,|\, X_\Phi\beta + \tilde{X}_\Phi B\gamma, \sigma^2 I_M) \;, \tag{4}$$

where ψ denotes additional parameters specifying the prior distributions on either γ or β. For example, a customary specification is

$$p(\psi, \sigma^2, \beta, \gamma) = \prod_{i=1}^{\tilde{P}} IG(\tau_i^2 \,|\, a_\tau, b_\tau) \times IG(\sigma^2 \,|\, a_\sigma, b_\sigma) \times N(\beta \,|\, \mu_\beta, V_\beta) \times N(\gamma \,|\, 0, \Delta_\psi) \;, \tag{5}$$

where $\psi = \{\tau_1^2, ..., \tau_{\tilde{P}}^2\}$ and Δ is $H\tilde{P} \times H\tilde{P}$ block-diagonal with j-th block given by $\tau_j^2 I_H$, for $j = 1, ..., \tilde{P}$. While (5) is a convenient choice for empirical investigations due to conjugate full conditional distributions, our method applies broadly to any basis function and any discrete or continuous mixture of Gaussian priors on the basis coefficients. In applications where the associations among the latent regression slopes is of importance, one could, for instance, adopt $p(\psi, \gamma) = IW(\psi \,|\, r, \Omega) \times N(\gamma \,|\, 0, \psi)$ with ψ as the $H\tilde{P} \times H\tilde{P}$ covariance matrix for γ. Our current focus is not, however, on such multivariate models, so we do not discuss them further except to note that (4) accommodates such extensions.

The likelihood in (4) is different from that by applying Φ to (3) because the error distribution in (4) is retained as the usual noise distribution without any effect of Φ. Hence, the model in (4) is a model analogous to (3) but applied to the *new* compressed data set $\{y_\Phi, X_\Phi, \tilde{X}_\Phi\}$. Working with a Φ-transformed model (3), where the distribution of the noise will be transformed according $\Phi\epsilon$, will not deliver the computational benefits, and is somewhat detrimental to the cause of data confidentiality (as in that case, the analyst need to know Φ) that are provided by (4).

For specifying Φ we pursue the idea of data oblivious Gaussian sketching (Sarlos, 2006), where we draw the elements of $\Phi = (\Phi_{ij})$ independently from $\mathrm{N}(0, 1/N)$ and fix them. The dominant computational operations for obtaining the sketched data using Gaussian sketches is $O(MN^2\tilde{P})$. While alternative computationally efficient data oblivious options such as the Hadamard sketch (Ailon & Chazelle, 2009) and the Clarkson-Woodruff sketch (Clarkson & Woodruff, 2017) are available for Φ, it is less pertinent in Bayesian settings since computation time of (4) far exceeds that for the sketching matrix. The compressed data serves as a surrogate for the Bayesian regression analysis with spatially varying coefficients. Since the number of compressed records is much smaller than the number of records in the uncompressed data

matrix, spatial model fitting becomes computationally efficient and economical in terms of storage as well as the number of floating point operations (flops). Importantly, original data are not recoverable from the compressed data, and the compressed data effectively reveal no more information than would be revealed by a completely new sample (Zhou et al., 2008). In fact, the original uncompressed data does not need to be stored or accessed at any stage in the course of the analysis.

2.1. Efficient Posterior Computation & Approximate Predictive Inference

In what follows, we discuss efficient computation offered by the data sketching framework. With prior distributions on parameters specified as in (5), posterior computation requires drawing Markov chain Monte Carlo (MCMC) samples sequentially from the full conditional posterior distributions of $\gamma|-$, $\beta|-$, $\sigma^2|-$ and $\tau_j^2|-$, $j = 1, \ldots, \tilde{P}$. To this end, $\sigma^2|- \sim IG(a_\sigma + M/2, b_\sigma + ||y_\Phi - X_\Phi\beta - \tilde{X}_\Phi B\gamma||^2/2)$, $\tau_j^2|- \sim IG(a_\tau + H/2, b_\tau + ||\gamma_j||^2/2)$ and $\beta|- \sim N\left(\left(X_\Phi^{\mathrm{T}}X_\Phi/\sigma^2 + I\right)^{-1} X_\Phi^{\mathrm{T}}(y_\Phi - \tilde{X}_\Phi B\gamma)/\sigma^2, \left(X_\Phi^{\mathrm{T}}X_\Phi/\sigma^2 + I\right)^{-1}\right)$ do not present any computational obstacles. The main computational bottleneck lies with $\gamma|-$,

$$N\left(\left(\frac{B^{\mathrm{T}}\tilde{X}_\Phi^{\mathrm{T}}\tilde{X}_\Phi B}{\sigma^2} + \Delta^{-1}\right)^{-1} B^{\mathrm{T}}\tilde{X}_\Phi^{\mathrm{T}}\frac{(y_\Phi - X_\Phi\beta)}{\sigma^2}, (B^{\mathrm{T}}\tilde{X}_\Phi^{\mathrm{T}}\tilde{X}_\Phi B/\sigma^2 + \Delta^{-1})^{-1}\right). \quad (6)$$

Efficient sampling of γ relies upon the Cholesky decomposition of $\left(B^{\mathrm{T}}\tilde{X}_\Phi^{\mathrm{T}}\tilde{X}_\Phi B/\sigma^2 + \Delta^{-1}\right)$ and solves triangular linear systems to draw a sample from (6). While numerically robust for small to moderately large H, computing and storing the Cholesky factor of this matrix involves $O((H\tilde{P})^3)$ and $O((H\tilde{P})^2)$ floating point operations, respectively (Golub & Van Loan, 2012). This results in computational and memory bottlenecks for a large number of basis functions, which may be required to estimate the spatial surface with sufficient local variation.

To achieve computational efficiency, we adapt a recent algorithm proposed in Bhattacharya et al. (2016) (in the context of ordinary linear regression with uncompressed data and small sample size) to our setting: (i) draw $\tilde{\gamma}_1 \sim N(0, \Delta)$ and $\tilde{\gamma}_2 \sim N(0, I_M)$; (ii) set $\tilde{\gamma}_3 = \tilde{X}_\Phi B\tilde{\gamma}_1/\sigma + \tilde{\gamma}_2$; (iii) solve $(\tilde{X}_\Phi B\Delta B^{\mathrm{T}}\tilde{X}_\Phi^{\mathrm{T}}/\sigma^2 + I_M)\tilde{\gamma}_4 = ((y_\Phi - X_\Phi\beta)/\sigma - \tilde{\gamma}_3)$; and (iv) set $\tilde{\gamma}_5 = \tilde{\gamma}_1 + \Delta B^{\mathrm{T}}\tilde{X}_\Phi^{\mathrm{T}}\tilde{\gamma}_4/\sigma$. The resulting $\tilde{\gamma}_5$ is a draw from the full conditional posterior distribution of γ. The computation is dominated by step (iii), which comprises $O(M^3 + M^2H\tilde{P})$. Finally, note that when basis functions involve parameters, they are updated using Metropolis-Hastings steps since no closed form full conditionals are generally available for them.

Predictive inference on $y(s_0)$ will proceed from the posterior predictive distribution

$$\mathbb{E}[p(y(s_0) \mid y_\Phi, \beta, \gamma, \sigma^2)] = \int p(y(s_0) \mid y_\Phi, \beta, \gamma, \sigma^2)p(\beta, \gamma, \sigma^2 \mid y_\Phi, \Phi)d\beta d\gamma d\sigma^2 \,, \quad (7)$$

where $\mathbb{E}[\cdot]$ is the expectation with respect to the posterior distribution in (4). This is easily achieved by drawing $y(s_0)^{(l)} \sim N(\sum_{p=1}^{P} x_p(s_0)\beta_p^{(l)} + \sum_{j=1}^{\tilde{P}} \tilde{x}_j(s_0)w_j(s_0)^{(l)}, \sigma^{2(l)})$ for each posterior sample $\{\beta^{(l)}, \gamma^{(l)}, \sigma^{2(l)}\}$ drawn from (4), where $w_j(s_0)^{(l)}$ is obtained from $\gamma^{(l)}$ using (2) and $l = 1, 2, \ldots, L$ indexes the L (post-convergence) posterior samples. The next section offers theoretical results related to the large sample consistency of the posterior distribution from the compressed varying coefficients model (4) and the posterior predictive distribution in (7) with respect to the probability law for the uncompressed oracle model in (1).

3. Posterior contraction from data sketching

3.1. Definitions and Notations

This section proves the posterior contraction properties of varying coefficients under the proposed framework. In what follows, we add a subscript N to the compressed response vector $y_{\Phi,N}$, compressed predictor matrix $\tilde{X}_{\Phi,N}$, dimension of the compression matrix M_N and the number of basis functions H_N to indicate that all of them increase with the sample size N. Naturally, the dimension of the basis coefficient vector γ and the compression matrix Φ are also functions of N, though we keep this dependence implicit. Since we do not assume a functional variable selection framework, we keep P fixed throughout, and not a function of N. We assume that $s_1, ..., s_N$ follow i.i.d. distribution G on $\mathcal{D}$ with G having a Lebesgue density g, which is bounded away from zero and infinity uniformly over $\mathcal{D}$. The true regression function is also given by (1), with the true varying coefficients $w_1^*(s), ..., w_{\tilde{P}}^*(s)$ belonging to the class of functions

$$\mathcal{F}_\xi(\mathcal{D}) = \{f : f \in L_2(\mathcal{D}) \cap \mathcal{C}^\xi(\mathcal{D}), E_{\mathcal{S}}[|f|] < \infty\}, \tag{8}$$

where $L_2(\mathcal{D})$ is the set of all square integrable functions on $\mathcal{D}$, $\mathcal{C}^\xi(\mathcal{D})$ is the class of at least ξ-times continuously differentiable functions in $\mathcal{D}$ and $E_{\mathcal{S}}$ denotes the expectation under the density of g. The probability and expectation under the true data generating model are denoted by P^* and E^*, respectively. For algebraic simplicity, we make a few simplifying assumptions in the model. To be more specific, we assume that $\beta = 0$ and $\sigma^2 = \sigma^{*2}$ is known and fixed at 1. The first assumption is mild since P does not vary with N and we do not consider variable selection. The second assumption is also customary in asymptotic studies (Vaart & Zanten, 2011). Furthermore, the theoretical results obtained by assuming σ^2 as a fixed value is equivalent to those obtained by assigning a prior with a bounded support on σ^2 (Van der Vaart et al., 2009).

For a vector $v = (v_1, ..., v_N)^{\mathrm{T}}$, we let $||\cdot||_1, ||\cdot||_2$ and $||\cdot||_\infty$ denote the L_1, L_2 and L_∞ norms, respectively, defined as $||v||_2 = (\sum_{n=1}^N v_n^2)^{1/2}$, $||v||_1 = \sum_{n=1}^N |v_n|$ and $||v||_\infty = \max_{n=1,..,N} |v_n|$, respectively. The number of nonzero elements in a vector is given by $||\cdot||_0$. In the case of a square integrable function $f(s)$ on $\mathcal{D}$, we denote the integrated L_2-norm of f by $||f||_2 = \left(\int_{\mathcal{D}} f(s)^2 g(s) ds\right)^{1/2}$ and the sup-norm of f by $||f||_\infty = \sup_{s\in\mathcal{D}} |f(s)|$. Thus $||\cdot||_\infty$ and $||\cdot||_2$ are used both for vectors and functions, and they should be interpreted based on the context. Finally, $e_{\min}(A)$ and $e_{\max}(A)$, respectively, represent the minimum and maximum eigenvalues of the square matrix A. The Frobenius norm of the matrix A is given by $||A||_F = \sqrt{\mathrm{tr}(A^{\mathrm{T}}A)}$. For two nonnegative sequences $\{a_N\}$ and $\{b_N\}$, we write $a_N \asymp b_N$ to denote $0 < \liminf_{N\to\infty} a_N/b_N \leq \limsup_{N\to\infty} a_N/b_N < \infty$. If $\lim_{N\to\infty} a_N/b_N = 0$, we write $a_N = o(b_N)$ or $a_N \prec b_N$. We use $a_N \lesssim b_N$ or $a_N = O(b_N)$ to denote that for sufficiently large N, there exists a constant $C > 0$ independent of N such that $a_N \leq C b_N$.

3.2. Assumption, Framework and Main Results

For simplicity, we assume $\Delta = I$ and that the random covariates $x_p(s)$, $p = 1, ..., P$ follow distributions which are independent of the distribution of the idiosyncratic error ϵ. We now state the following assumptions on the basis functions, H_N, M_N, covariates and the sketching or compression matrix.

(A) For any $w_j^*(s) \in \mathcal{F}_\xi(\mathcal{D})$, there exists γ_j^* such that $||w_j^* - B_j^{\mathrm{T}}\gamma_j^*||_\infty = \sup_{s\in\mathcal{D}} |w_j^*(s) - \sum_{h=1}^{H_N} B_{jh}(s)\gamma_{jh}^*| = O(H_N^{-\xi})$, for $j = 1, ..., \tilde{P}$, and $||\gamma^*||_2^2 \prec M_N^{1/(1+\xi)}$.

(B) N, M_N, H_N satisfy $M_N = o(N)$ and $H_N \asymp M_N^{1/(2\xi+2)}$.

(C) $||\Phi\Phi^{\mathrm{T}} - I_{M_N}||_F \leq C'\sqrt{M_N/N}$, for some constant $C' > 0$, for all large N.

(D) The random covariate $x_p(s)$ are uniformly bounded for all $s \in \mathcal{D}$, and w.l.g., $|x_p(s)| \leq 1$, for all $p = 1, ..., P$ and for all $s \in \mathcal{D}$.

(E) There exists a sequence κ_N such that $||\tilde{X}_{\Phi,N}\alpha||^2 \asymp \kappa_N ||\tilde{X}_N \alpha||^2$, such that $1 \prec N\kappa_N \prec M_N$ for any vector $\alpha \in \mathbb{R}^{N\tilde{P}}$.

Assumption (A) holds for orthogonal Legendre polynomials, Fourier series, B-splines and wavelets (Shen & Ghosal, 2015). Assumption (B) provides an upper bound on the growth of M_N and H_N as a function of N. Assumption (C) is a mild assumption based on the theory of random matrices and occurs with probability at least $1 - e^{-C''M_N}$ when Φ is constructed using the Gaussian sketching for a constant $C'' > 0$ (see Lemma 5.36 and Remark 5.40 of Vershynin (2010)). Assumption (D) is a technical condition customarily used in functional regression analysis (Bai et al., 2019). Finally, Assumption (E) characterises the class of feasible compression matrices, roughly explaining how the linear structure of the columns of the original predictor matrix is related to that of the compressed predictor matrix. Such an assumption is reasonable for the set of random compression matrices for a sequence κ_N depending on N, M_N and $\tilde{P}$ (Ahfock et al., 2017).

Let $w(s) = (w_1(s), ..., w_{\tilde{P}}(s))^{\mathrm{T}}$ and $w^*(s) = (w_1^*(s), ..., w_{\tilde{P}}^*(s))^{\mathrm{T}}$ be the $\tilde{P}$-dimensional fitted and true varying coefficients. Let $\|w - w^*\|_2 = \sum_{j=1}^{\tilde{P}} \|w_j - w_j^*\|_2$ denote the sum of integrated L_2 distances between the true and the fitted varying coefficients. Define the set $\mathcal{C}_N = \left\{ w : ||w - w^*||_2 > \tilde{C}\theta_N \right\}$, for some constant $\tilde{C}$ and some sequence $\theta_N \to 0$ and $M_N\theta_N^2 \to \infty$. Further suppose $\pi_N(\cdot)$ and $\Pi_N(\cdot)$ are the prior and posterior densities of w with N observations, respectively. From equation (2), the prior distribution on w is governed by the prior distribution on γ, so that the posterior probability of $\mathcal{C}_N$ can be expressed as,

$$\Pi_N(\mathcal{C}_N | y_{\Phi,N}, \tilde{X}_{\Phi,N}) = \frac{\int_{\mathcal{C}_N} f(y_{\Phi,N}|\tilde{X}_{\Phi,N}, \gamma)\pi_N(\gamma)}{\int f(y_{\Phi,N}|\tilde{X}_{\Phi,N}, \gamma)\pi_N(\gamma)},$$

where $f(y_{\Phi,N}|\tilde{X}_{\Phi,N}, \gamma)$ is the joint density of $y_{\Phi,N}$ under model (4). We begin with the following important result from the random matrix theory.

LEMMA 1. *Consider the $M_N \times N$ compression matrix Φ with each entry being drawn independently from $N(0, 1/N)$. Then, almost surely*

$$(\sqrt{N} - \sqrt{M_N} - o(\sqrt{N}))^2/N \leq e_{\min}(\Phi\Phi^{\mathrm{T}}) \leq e_{\max}(\Phi\Phi^{\mathrm{T}}) \leq (\sqrt{N} + \sqrt{M_N} + o(\sqrt{N}))^2/N, \tag{9}$$

when both $M_N, N \to \infty$.

Proof. This is a consequence of Theorem 5.31 and Corollary 5.35 of Vershynin (2010). □

The inequalities in (9) is used to derive the following two results, which we present as Lemma 2 and 3.

LEMMA 2. *Let P^* denote the true probability distribution of y_N and $f^*(y_{\Phi,N}|\gamma^*)$ denotes the density of $y_{\Phi,N}$ (omitting explicit dependence on $\tilde{X}_{\Phi,N}$) under the true data generating model. Define*

$$\mathcal{A}_N = \left\{ y : \int \left\{ f(y_{\Phi,N}|\gamma)/f^*(y_{\Phi,N}|\gamma^*) \right\} \pi_N(\gamma) d\gamma \leq \exp(-CM_N\theta_N^2) \right\}. \tag{10}$$

Then $P^*(\mathcal{A}_N) \to 0$ as $M_N, N \to \infty$ for any constant $C > 0$.

Proof. See Appendix. □

LEMMA 3. *Let γ^* be any fixed vector in the support of γ and let $\mathcal{B}_N = \{\gamma : ||\gamma - \gamma^*||_2 \leq C_{2w}\theta_N H_N^{1/2}\}$ for some constant $C_{2w} > 0$. Then there exists a sequence ζ_N of random variables depending on $\{y_{\Phi,N}, X_{\Phi,N}\}$ and taking values in $(0,1)$ such that*

$$\mathbb{E}^*(\zeta_N) \lesssim \exp(-M_N\theta_N^2) \text{ and } \sup_{\gamma \in \mathcal{B}_N^c} \mathbb{E}_\gamma(1-\zeta_N) \lesssim \exp(-M_N\theta_N^2), \tag{11}$$

where $\mathbb{E}_\gamma$ and $\mathbb{E}^*$ denote the expectations under the distributions $f(\cdot \,|\, \gamma)$ and $f^*(\cdot \,|\, \gamma^*)$, respectively.

Proof. See Appendix. □

We use the above results to establish the posterior contraction result for the proposed model.

THEOREM 1. *Under Assumptions (A)-(E), our proposed model (4) satisfies $\max_{j=1,\ldots,\tilde{P}} \sup_{w_j^* \in \mathcal{F}_\xi(\mathcal{D})} \mathbb{E}^*\Pi_N(\mathcal{C}_N \,|\, y_{\Phi,N}, \tilde{X}_{\Phi,N}) \to 0$, as $N, M_N \to \infty$ and with the posterior contraction rate $\theta_N \asymp M_N^{-\xi/(2\xi+2)}$.*

Proof. See Appendix. □

Since $\theta_N \to 0$ as $N \to \infty$, the model consistently estimates the true varying coefficients under the integrated L_2-norm. Further, data compression decreases the effective sample size from N to M_N, hence, the contraction rate θ_N obtained in Theorem 1 is optimal and adaptive to the smoothness of the true varying coefficients. Our next theorem justifies the two-stage prediction strategy described in Section 2.1.

THEOREM 2. *For any location s_0 drawn randomly with the density g and corresponding predictors $\tilde{x}_1(s_0), \ldots, \tilde{x}_{\tilde{P}}(s_0)$, let f_u be the predictive density $p(y(s_0) \,|\, \tilde{x}_1(s_0), \ldots, \tilde{x}_{\tilde{P}}(s_0), w(s_0))$ derived from (1) without data compression. Let f^* be the true data generating model (i.e., (1) with $w(s_0)$ fixed at $w^*(s_0)$). Given s_0 and $\tilde{x}_1(s_0), \ldots, \tilde{x}_{\tilde{P}}(s_0)$, define $h(f_u, f^*) = \int(\sqrt{f_u} - \sqrt{f^*})^2$ as the Hellinger distance between the densities f_u and f^*. Then*

$$\mathbb{E}^*\mathbb{E}\mathbb{E}_{\mathcal{S}}[h(f_u, f^*) \,|\, \tilde{X}_{\Phi,N}, y_{\Phi,N}] \to 0, \text{ as } N, M_N \to \infty, \tag{12}$$

where $\mathbb{E}_{\mathcal{S}}$, $\mathbb{E}$ and $\mathbb{E}^*$ stand for expectations with respect to the density g, the posterior density $\Pi_N(\cdot|\tilde{X}_{\Phi,N}, y_{\Phi,N})$ and the true data generating distribution, respectively.

Proof. See Appendix. □

The theorem states that the predictive density of the VCM model in (1) is arbitrarily close to the true predictive density even when we plug-in inference on parameters from (4).

4. SIMULATION RESULTS

4.1. *Inferential performance*

We empirically validate our proposed approach using (4), henceforth abbreviated as *geoS*, by comparing its inferential performance and computational efficiency with the uncompressed model (3) on some simulated data. We simulate data by using a fixed set of spatial locations $s_1, \ldots, s_N$ that were drawn uniformly over the domain $\mathcal{D} = [0,1] \times [0,1]$. We set $\tilde{P} = P = 3$ and assume $\beta = 0$, i.e., all predictors have purely space-varying coefficients. We set $\tilde{x}_1(s_i) = 1$, for all $i = 1, \ldots, N$, while the values of $\tilde{x}_j(s_1), \ldots, \tilde{x}_j(s_N)$ for $j = 2, 3$ were set to independently values from $N(0,1)$. For each $n = 1, \ldots, N$, the response $y(s_n)$ is drawn independently from $N(w_1^*(s_n) + w_2^*(s_n)\tilde{x}_2(s_n) + w_3^*(s_n)\tilde{x}_3(s_n), \sigma^{*2})$ following (3), where σ^{*2} is set to be 0.1. The true space-varying coefficients ($w_j^*(s)$s) are simulated from a Gaussian process with mean 0 and covariance kernel $C(\cdot,\cdot;\theta_j)$, i.e., $(w_j^*(s_1), ..., w_j^*(s_N))^{\mathrm{T}}$ is drawn from $N(0, C^*(\theta_j))$, for each $j = 1, \ldots, \tilde{P}$, where $C^*(\theta_j)$ is an $N \times N$ matrix with the (n, n')th element $C(s_n, s_{n'}; \theta_j)$. We set the covariance kernel $C(\cdot,\cdot;\theta_j)$ to be the exponential covariance

function given by

$$C(s, s'; \theta_j) = \delta_j^2 \exp\left\{-\frac{1}{2}\left(\frac{||s - s'||}{\phi_j}\right)\right\}, \ j = 1, 2, 3, \tag{13}$$

with the true values of $\delta_1^2, \delta_2^2, \delta_3^2$ set to $1, 0.8, 1.1$, respectively. We fix the true values of ϕ_1, ϕ_2, ϕ_3 at $1, 1.25, 2$, respectively.

While fitting *geoS* and its uncompressed analogue (3), the varying coefficients are modelled through the linear combination of H basis functions as in (2), where these basis functions are chosen as the tensor-product of B-spline bases of order $q = 4$ (Shen & Ghosal, 2015). More specifically, for $s = (s^{(1)}, s^{(2)})$, the j-th varying coefficient is modelled as

$$w_j(s) = \sum_{h_1=1}^{H_1} \sum_{h_2=1}^{H_2} B_{jh_1}^{(1)}(s^{(1)}) B_{jh_2}^{(2)}(s^{(2)}) \gamma_{jh_1h_2} \ , \tag{14}$$

where the marginal B-splines $B_{jh_1}^{(1)}$, $B_{jh_2}^{(2)}$ are defined on sets of H_1 and H_2 knots, respectively. The knots are chosen to be equally-spaced so the entire set of $H = H_1H_2$ knots is uniformly spaced over the domain $\mathcal{D}$. We complete the hierarchical specification by assigning independent $IG(2, 0.1)$ priors (mean 0.1 with infinite variance) for σ^2 and τ_j^2 for each $j = 1, \ldots, P$.

We implemented our models in the `R` statistical computing environment on a Dell XPS 13 PC with Intel Core i7-8550U CPU @ 4.00GHz processors at 16 GB of RAM. For each of our simulation datasets we ran a single-threaded MCMC chain for 5000 iterations. Posterior inference was based upon 2000 samples retained after adequate convergence was diagnosed using Monte Carlo standard errors and effective sample sizes (ESS) using the `mcmcse` package in `R`. All source codes for these experiments are available from `https://github.com/LauraBaracaldo/Spatial-Meta-Kriging-for-Distributed-Inference-for-Binary-Response`.

Table 1 summarises the estimates of varying coefficients and the predictive performance for *geoS* in comparison to the uncompressed model. We applied these models to data generated with $N = 5000$ (case 1) and $N = 10000$ (case 2). For both cases the compressed dimension is taken to be $M \approx 10\sqrt{N}$ which seems to be effective from empirical considerations in our simulations. We provide further empirical justification for this choice in Section 4.2. Our *geoS* approach compresses the sample sizes to $M = 700$ and $M = 1000$ in cases 1 and 2, respectively. The number of fitted basis functions in cases 1 & 2 are $H = 225, 256$, respectively.

Figures 1 and 2 present the estimated varying coefficients by *geoS* and the uncompressed data model for cases 1 and 2, respectively. These figures reveal practically indistinguishable point estimates offered by *geoS* and the uncompressed model. The mean squared error of estimating varying coefficients, defined as $\sum_{j=1}^{3}\sum_{n=1}^{N}(\widehat{w}_j(s_n) - w_j^*(s_n))^2/(3N)$ (where $\widehat{w}_j(s_n)$ is the posterior median of $w_j(s_n)$), also confirms very similar point estimates offered by the compressed and uncompressed models (see Table 1). Further, *geoS* offers close to nominal coverage for 95% credible intervals for varying coefficients, with little wider credible intervals compared to uncompressed data model. This can be explained by the smaller sample size for the *geoS* model, though the difference turns out to be minimal. We also carry out predictive inference using *geoS* (Section 2.1). Table 1 presents mean squared predictive error (MSPE), average length and coverage for the 95% predictive intervals, based on $N^* = 500$ out of the sample observations. We find *geoS* delivers posterior predictive estimates and predictive coverage that are very consistent with the uncompressed model, perhaps with marginally wider predictive intervals than those without compression.

Finally, the computational efficiency of both models are computed based on the metric $\log_2(ESS/\text{Computation Time})$, where ESS denotes the effective sample size averaged over the MCMC samples of all parameters. We find *geoS* is almost 270% and 223% more efficient than the uncompressed model for $N = 5,000$ and $N = 10,000$, respectively, while delivering almost indistinguishable substantive inference on the spatial effects.

	$N = 5000$, $H = 225$		$N = 10000$, $H = 256$	
	(geoS) $M = 700$	*Uncompressed*	*(geoS)* $M = 1000$	*Uncompressed*
MSE (SVC)	0.0474	0.0168	0.0429	0.0178
95% CI length	0.8368	0.6182	0.7222	0.5531
95% CI Coverage	0.9448	0.9322	0.9153	0.9026
MSPE	0.2574	0.1833	0.2283	0.1605
95% PI length	1.9717	1.5168	1.8613	1.5148
95% PI coverage	0.936	0.925	0.954	0.930
Computation efficiency	2.2050	0.8079	0.9755	0.4356

Table 1: Results for simulation cases 1 & 2 for the compressed *geoS* and uncompressed models. Mean Squared Error (MSE), length and coverage of 95% CI for the spatially varying coefficients. We also present mean squared prediction error (MSPE), coverage and length of 95% predictive intervals for the competing models. Computation efficiency for the *geoS* with the uncompressed data model is also recorded.

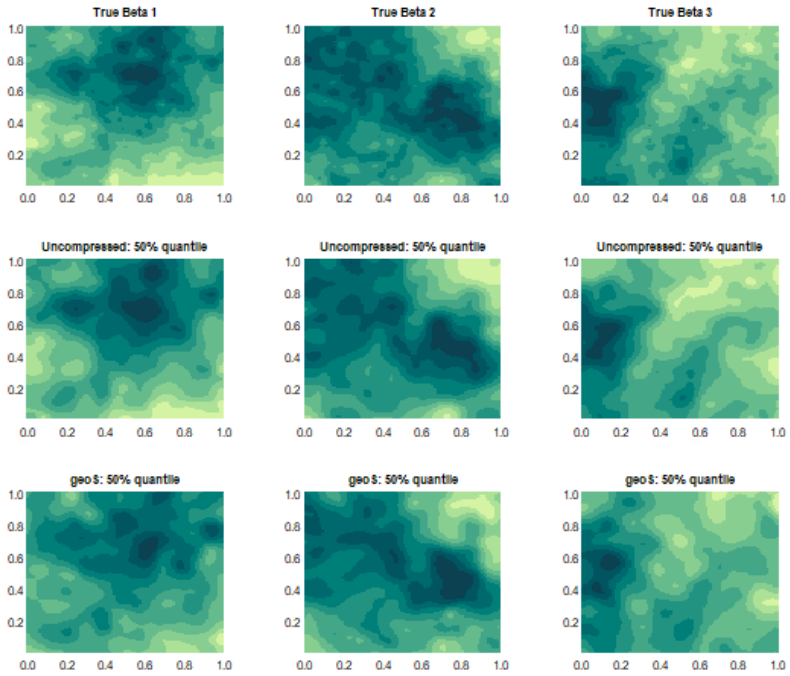


Figure 1: Simulation case 1: $(N, H) = (5000, 225)$. Two-dimensional true and predicted surfaces over the unit square $\mathcal{D} = [0, 1] \times [0, 1]$. First row corresponds to the surfaces of true space-varying coefficients $\beta_p^*(s)$, $p = 1, 2, 3$. Rows 2 and 3 correspond to the predicted 50% quantile surfaces for the uncompressed and compressed *geoS* models respectively.

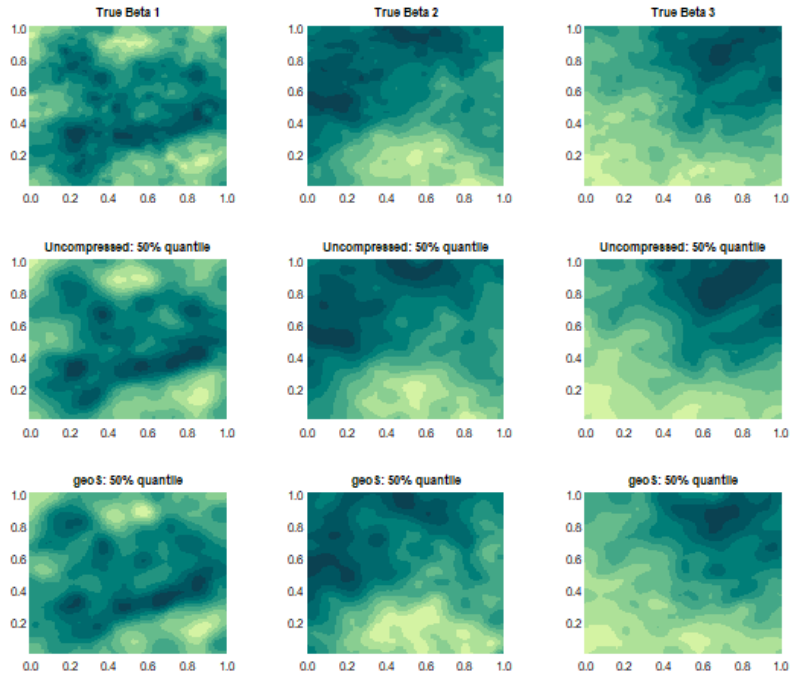


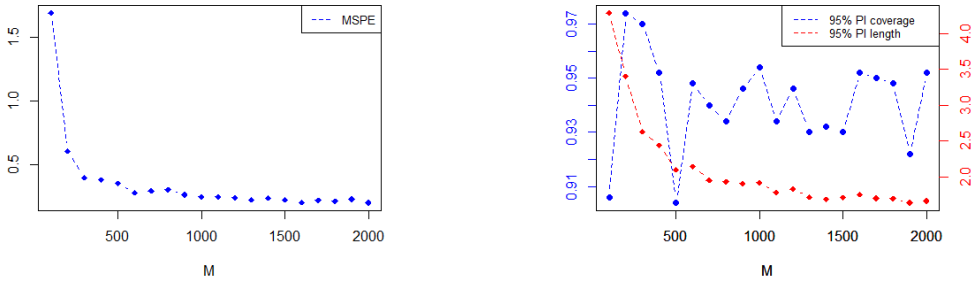
Figure 2: Simulation case 2: $(N, H) = (10000, 256)$. Two-dimensional true and predicted surfaces over the unit square $\mathcal{D} = [0, 1] \times [0, 1]$. First row corresponds to the surfaces of true space-varying coefficients $\beta_p^*(s)$, $p = 1, 2, 3$. Rows 2 and 3 correspond to the predicted 50% quantile surfaces for the uncompressed and compressed *geoS* models respectively.

4.2. *Choice of the dimension of the compression matrix M*

We present investigations into the choice of the appropriate compression matrix size M. For simulated data with sample size $N = 10000$, we ran our model for different values of $M = k\sqrt{N}$, $k = 1, \ldots, 20$. Figure (3) shows the variations in point-wise and interval prediction reflected in the $MSPE$ and 95% predicted interval coverage and length, respectively. Unsurprisingly, as M increases the MSPE drops with a diminished rate of decline until the $k \sim 10$. In terms of interval prediction, predictive coverage seems to oscillate within the narrow interval $(0.9, 0.97)$ for all values of M, but the length of the predictive interval improves as M increases and starts to stabilise at around $k \sim 10$. We observe that the choice of $M \sim 10\sqrt{N}$ leads to good performance across various simulations and real data analysis.

5. Vegetation Data Analysis

We implement *geoS* to analyse vegetation data gathered through the Moderate Resolution Imaging Spectroradiometer (MODIS), which resides aboard the Terra and Aqua platforms on NASA spacecrafts. MODIS vegetation indices, produced on 16-day intervals and at multiple spatial resolutions, provide consistent information on the spatial distribution of vegetation canopy greenness, a composite property of leaf area, chlorophyll and canopy structure. The variable of interest will be the Normalised Difference Vegetation Index (NDVI), which quantifies the relative vegetation density for each pixel in a satellite image, by measuring the difference between the reflection in the near-infrared spectrum (NIR) and the red light reflection (RED):



(a) (b)

Figure 3: (a) MSPE, (b) 95% predictive interval coverage and length for different choices of M

$NDVI = \frac{NIR-RED}{NIR+RED}$. High NDVI values, ranging between 0.6 and 0.9 indicate high density of green leaves and healthy vegetation, whereas low values, 0.1 or below, correspond to low or absence of vegetation as in the case of urbanised areas. When analysed over different locations, NDVI can reveal changes in vegetation due to human activities such as deforestation and natural phenomena such as wild fires and floods.

Our analysis will be focused on geographical data that was mapped on a sinusoidal (SIN) projected grid, located on the western coast of the United States, more precisely zone *h08v05*, between $30°N$ to $40°N$ latitude and $104°W$ to $130°W$ longitude (see Figure 4(a)). The data set, which was downloaded using the `R` package MODIS, comprises $133,000$ observed locations where the response was measured through the MODIS tool over a 16-day period in April, 2016. We retained $N = 113,000$ observations (randomly chosen) for model fitting and held out the rest for prediction. In order to fit (1), we set $y(s_n)$ to be the transformed NDVI $(\log(NDVI) + 1)$, $P = \tilde{P} = 2$ and consider the $P \times 1$ vector of predictors that includes an intercept and a binary index of urban area, both with fixed effects and spatially varying coefficients, i.e., $x(s_n) = \tilde{x}(s_n) = (1,\ x_2(s_n))^{\mathrm{T}}$, with $x_2(s_n) = \mathbb{1}_U(s_n)$, where U denotes an urban area.

As in Section 4, we fit *geoS* with $M \sim 10\sqrt{N} = 2300$ and its uncompressed counterpart (3), by modelling the varying coefficients through a linear combination of basis functions constructed using the tensor-product of B-splines of order $q = 4$ as in (14). We set the number of knots $H = H_1H_2 = 39^2 = 1521$ to be uniformly distributed over the domain $\mathcal{D}$, which results in $HP = 3042$ basis coefficients γ_{jh} that are estimated. Specification of priors are identical to the simulation studies for σ^2, and τ_j^2, $j = 1, ..., P$; for β_j, $j = 1, ..., P$ we set a flat prior.

We ran an MCMC chain for 5000 iterations and retained 2000 samples for posterior inference after adequate convergence was diagnosed. The posterior mean of β_1 and β_2, along with their estimated 95% credible intervals corresponding to *geoS* and the uncompressed model are presented in Table 2. Additionally, Table 2 offers predictive inference from both competitors based on $N^* = 20,000$ test observations. According to both models there is a global pattern of relatively low vegetation density for areas with positive urban index as the estimated slope coefficient β_2 is negative in the compressed *geoS* and in the uncompressed models. In terms of point prediction and quantification of predictive uncertainty, the two competitors offer practically indistinguishable results, as revealed by Table 2. Further, Figure 4 shows that the 2.5%, 50% and 97.5% quantiles for the posterior predictive distribution are almost identical for the two competitors across the spatial domain, with the exception of neighbourhoods around locations having

lower NDVI values. Notably, *geoS* offers nominal coverage for 95% prediction intervals, even with a significant reduction in the sample size from $N = 113,000$ to $M = 2300$. Data sketching to such a scale considerably reduces the computation time, leading to a much higher computation efficiency of *geoS* in comparison with its uncompressed analogue.

	(geoS) $M = 2300$	*Uncompressed*
β_1	0.222 (0.212, 0.230)	0.229 (0.219, 0.237)
β_2	-0.060 (-0.074, -0.047)	-0.071 (-0.082, -0.059)
$MSPE$	0.00327	0.00276
95% PI length	0.23445	0.22136
95% PI coverage	0.95250	0.95411
Computation efficiency	3.5424	0.46901

Table 2: Median and 95% credible interval of β_1, β_2 for geoS and its uncompressed analogue are presented for the Vegetation data analysis. We also present MSPE, coverage and length of 95% predictive intervals for the competing models. Computational efficiency for the two competing models are also provided.

6. SUMMARY

We have developed Bayesian sketching for spatially oriented data using spatial regression models. The method achieves dimension reduction by compressing the data using a random linear transformation. The approach is different to the prevalent methods for large spatial data in that no new models or algorithms need to be developed since those available for existing spatially varying regression models can be directly applied to the compressed data. We establish attractive concentration properties of the posterior and posterior predictive distributions and empirically demonstrate the effectiveness of this method for analysing large spatial data sets. Access to the values of the response and predictors in the full data are not required at stage of inference, which preserves data confidentiality should that be of concern in the application.

7. ACKNOWLEDGEMENTS

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APPENDIX

This section contains theoretical results building up to the proofs of Theorems 1 and 2. Lemma 1 states an important result from random matrix theory that is easily obtained from Theorem 5.31 and Corollary 5.35 of Vershynin (2010). We prove Lemmas 2 and 3. The results in Lemma 1-3 are further used to prove Theorems 1 and 2.

Proof of Lemma 2

Proof. Define

$$\mathcal{A}_{1N} = \left\{K(f^*, f) \leq M_N\theta_N^2,\ V(f^*, f) \leq M_N\theta_N^2\right\}. \tag{15}$$

By Lemma 10 in Ghosal et al. (2007), to show (10) it is enough to show that $\Pi(\mathcal{A}_{1N}) \gtrsim \exp(-C_2 M_N\theta_N^2)$,

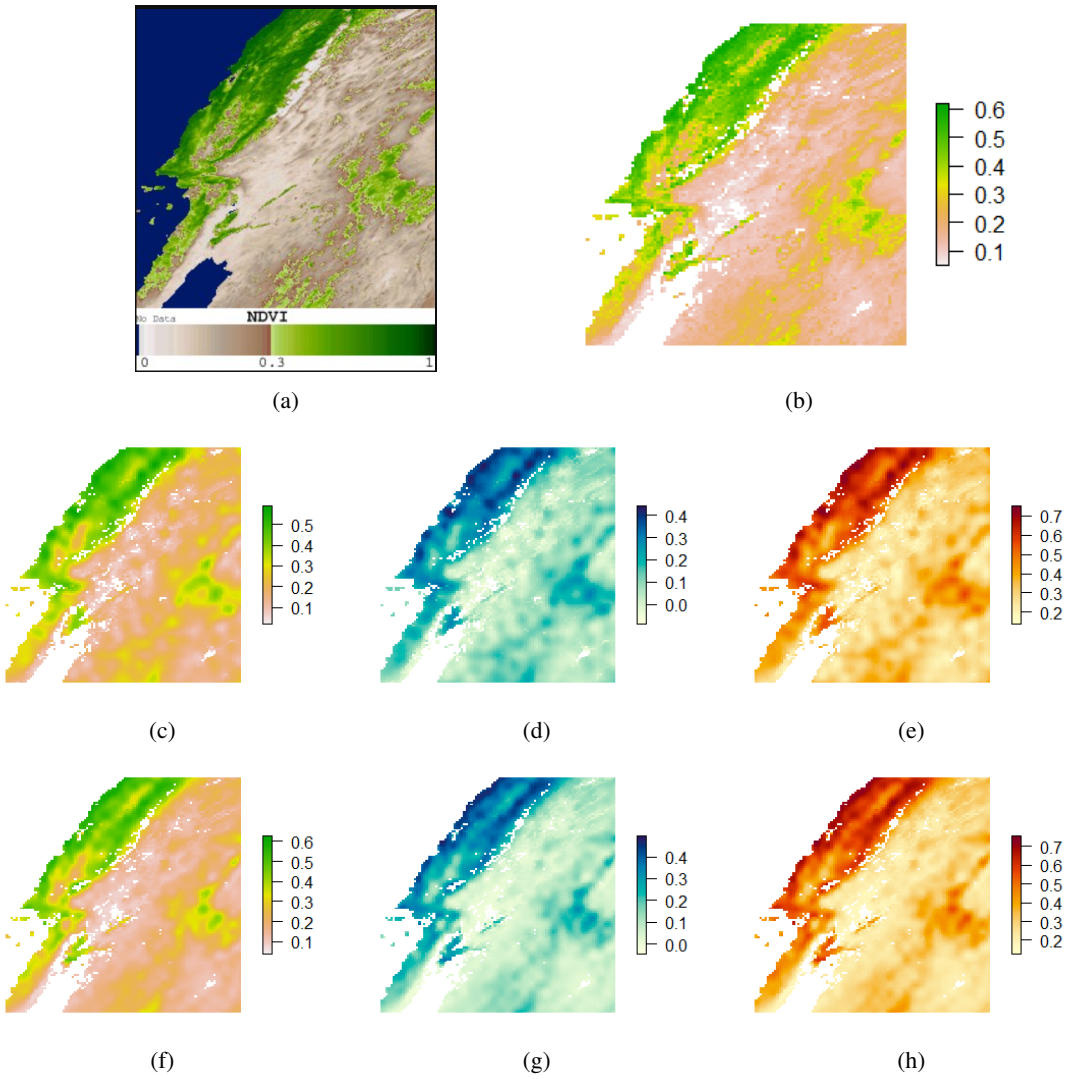

Figure 4: Coloured NDVI images of western United States (zone h08v05). (a) Satellite image: MODIS/Terra Vegetation Indices 16-Day L3 Global 1 km SIN Grid - 2016.04.06 to 2016.04.21; (b) True NDVI surface (raw data). Figures (c), (d) & (e) present NVDI predicted 50%, 2.5% and 97.5% quantiles for the *geoS* model. Figures (f), (g) & (h) present NVDI Predicted 50%, 2.5% and 97.5% quantiles for the uncompressed model.

for some constant $C_2 > 0$. Let e_k, $1 \leq k \leq M_N$ be the ordered eigenvalues of $(\Phi\Phi^{\mathrm{T}})^{-1}$. After some calculations, we derive the following expressions,

$$K(f^*, f) = \frac{1}{2}\left\{\sum_{k=1}^{M_N}(e_k - 1 - \log(e_k)) + \mathbb{E}_{\mathcal{S}}\mathbb{E}_{\mathcal{X}}\left[||\tilde{X}_{\Phi,N}B(\gamma - \gamma^*) - \tilde{X}_{\Phi,N}\eta^*||_2^2\right]\right\} \text{ and}$$

$$V(f^*, f) = \sum_{k=1}^{M_N}\frac{(1-e_k)^2}{2} + \mathbb{E}_{\mathcal{S}}E_{\mathcal{X}}\left[||(\Phi\Phi^{\mathrm{T}})^{-1}(\tilde{X}_{\Phi,N}B(\gamma - \gamma^*) - \tilde{X}_{\Phi,N}\eta^*)||_2^2\right], \qquad (16)$$

where $\eta^* = (\eta^*(s_1)^{\mathrm{T}}, ..., \eta^*(s_N)^{\mathrm{T}})^{\mathrm{T}}$, $\eta^*(s) = (\eta_1^*(s), \ldots, \eta_{\tilde{P}}^*(s))^{\mathrm{T}}$, $\eta_j^*(s) = w_j^*(s) - \sum_{h=1}^{H_N} B_{jh}(s)\gamma_{jh}^*$. Expanding $\log(e_k)$ in the powers of $(1-e_k)$ and using Lemma 1 in Jeong & Ghosal (2020) we find $(e_k - 1 - \log(e_k)) \sim (1-e_k)^2/2$. Another use of Lemma 1 in Jeong & Ghosal (2020) yields $\sum_{k=1}^{M_N}(1-e_k)^2 \lesssim ||I - \Phi\Phi^{\mathrm{T}}||_F^2 \lesssim M_N/N \leq M_N\theta_N^2$. Using Lemma 1, $e_k \asymp 1$ for all $k = 1, ..., M_N$. Hence, from (16)

$$\begin{aligned}\Pi(\mathcal{A}_{1N}) &\gtrsim \Pi\left(\left\{\gamma : \mathbb{E}_{\mathcal{S}}\mathbb{E}_{\mathcal{X}}\left[||\tilde{X}_{\Phi,N}B(\gamma-\gamma^*) - \tilde{X}_{\Phi,N}\eta^*||_2^2\right] \lesssim M_N\theta_N^2\right\}\right) \\ &\geq \Pi\left(\left\{\gamma : \mathbb{E}_{\mathcal{S}}\mathbb{E}_{\mathcal{X}}\left[||\tilde{X}_{\Phi,N}B(\gamma-\gamma^*)||_2^2\right] + \mathbb{E}_{\mathcal{S}}\mathbb{E}_{\mathcal{X}}\left[||\tilde{X}_{\Phi,N}\eta^*||_2^2\right] \lesssim M_N\theta_N^2/2\right\}\right),\end{aligned} \tag{17}$$

where we use $||a-b||_2^2 \leq 2(||a||_2^2 + ||b||_2^2)$, for all $a, b \in \mathbb{R}$. Let $B_j(s_n) = (B_{j1}(s_n), ..., B_{jH_N}(s_n))^{\mathrm{T}}$, for $n = 1, ..., N$ and $j = 1, ..., \tilde{P}$. By Assumption (E),

$$\begin{aligned}\mathbb{E}_{\mathcal{S}}\mathbb{E}_{\mathcal{X}}\left[||\tilde{X}_{\Phi,N}B(\gamma-\gamma^*)||_2^2\right] &\asymp \kappa_N E_{\mathcal{S}}\mathbb{E}_{\mathcal{X}}\left[||\tilde{X}_N B(\gamma-\gamma^*)||_2^2\right] \\ &= \kappa_N(\gamma-\gamma^*)^{\mathrm{T}}\mathbb{E}_{\mathcal{S}}\mathbb{E}_{\mathcal{X}}\left[B^{\mathrm{T}}\tilde{X}_N^{\mathrm{T}}\tilde{X}_N B\right](\gamma-\gamma^*).\end{aligned}$$

Recalling that $B^{\mathrm{T}}\tilde{X}_N^{\mathrm{T}}\tilde{X}_N B$ is a $H_N\tilde{P} \times H_N\tilde{P}$ matrix with the (j,j')-th block given by $\sum_{n=1}^N \tilde{x}_j(s_n)B_j(s_n)B_{j'}(s_n)^{\mathrm{T}}\tilde{x}_{j'}(s_n)$, we obtain

$$\begin{aligned}\mathbb{E}_{\mathcal{S}}E_{\mathcal{X}}\left[\sum_{n=1}^N \tilde{x}_j(s_n)B_j(s_n)B_{j'}(s_n)^{\mathrm{T}}\tilde{x}_{j'}(s_n)\right] &\asymp \mathbb{E}_{\mathcal{S}}\left[\sum_{n=1}^N B_j(s_n)B_{j'}(s_n)^{\mathrm{T}}\right] \\ &= N\mathbb{E}_{\mathcal{S}}\left[B_j(s_1)B_{j'}(s_1)^{\mathrm{T}}\right],\end{aligned}$$

where the last equation follows since $s_1, ..., s_N$ are i.i.d.. Hence,

$$\mathbb{E}_{\mathcal{S}}\mathbb{E}_{\mathcal{X}}\left[||\tilde{X}_{\Phi,N}B(\gamma-\gamma^*)||_2^2\right] \asymp N\kappa_N\mathbb{E}_{\mathcal{S}}\left[||B(s_1)(\gamma-\gamma^*)||_2^2\right] \asymp N\kappa_N||\gamma-\gamma^*||_2^2/H_N, \tag{18}$$

where $B(s) = [B_1(s) : \cdots : B_{\tilde{P}}(s)]^{\mathrm{T}}$. The last expression follows from Lemma A.1 of Huang et al. (2004). From Assumption (E) again,

$$\begin{aligned}\mathbb{E}_{\mathcal{S}}\mathbb{E}_{\mathcal{X}}\left[||\tilde{X}_{\Phi,N}\eta^*||_2^2\right] \asymp \kappa_N\mathbb{E}_{\mathcal{S}}\mathbb{E}_{\mathcal{X}}\left[||\tilde{X}_N\eta^*||_2^2\right] &= \kappa_N\mathbb{E}_{\mathcal{S}}\mathbb{E}_{\mathcal{X}}\left[\sum_{n=1}^N\sum_{j=1}^{\tilde{P}}\tilde{x}_j(s_n)^2\eta_j^*(s_n)^2\right] \\ &\asymp \kappa_N\mathbb{E}_{\mathcal{S}}\left[\sum_{n=1}^N\sum_{j=1}^{\tilde{P}}\eta_j^*(s_n)^2\right] \lesssim N\kappa_N H_N^{-2\xi},\end{aligned} \tag{19}$$

where the last inequality follows from Assumption (A). From (17),

$$\begin{aligned}\Pi(\mathcal{A}_{1N}) &\gtrsim \Pi\left(\gamma : N\kappa_N||\gamma-\gamma^*||_2^2/H_N + N\kappa_N H_N^{-2\xi} \lesssim M_N\theta_N^2/2\right) \\ &\gtrsim \Pi\left(\gamma : N\kappa_N||\gamma-\gamma^*||_2^2 \leq M_N H_N\theta_N^2\right),\end{aligned}$$

where the last step follows from Assumptions (B) and (E). Using the fact that $\int_a^b \exp(-x^2/2)dx \geq \exp(-(a^2+b^2)/2)(b-a)$, we obtain

$$\Pi\left(\gamma : N\kappa_N||\gamma-\gamma^*||_2^2 \leq M_N H_N \theta_N^2\right) \geq \prod_{h,j=1}^{H_N,\tilde{P}} \Pi(|\gamma_{jh}-\gamma^*_{jh}| \leq \theta_N/\sqrt{\tilde{P}})$$
$$\geq \exp(-||\gamma^*||_2^2 - \theta_N^2 H_N)(2\theta_N/\sqrt{\tilde{P}})^{H_N\tilde{P}} \gtrsim \exp(-M_N\theta_N^2 C_2),$$

for any $C_2 > 0$, where the first inequality follows from Assumption (E) and the last inequality follows from $H_N P \log(\sqrt{\tilde{P}}/2\theta_N) \prec M_N\theta_N^2$ (since $M_N\theta_N^2 \asymp M_N^{1/(1+\xi)}$ while $H_N \prec M_N^{1/(1+\xi)}$). □

Proof of Lemma 3

Proof. Denote $\tilde{X}_{\Phi,B,N} = \tilde{X}_{\Phi,N}B$, $\widehat{\gamma} = (\tilde{X}^{\mathrm{T}}_{\Phi,B,N}\tilde{X}_{\Phi,B,N})^{-1}\tilde{X}^{\mathrm{T}}_{\Phi,B,N}y_{\Phi,N}$ and a sequence of random variables $\zeta_N = I(||\tilde{X}_{\Phi,B,N}\widehat{\gamma} - \tilde{X}_{\Phi,B,N}\gamma^*||_2 \gtrsim \theta_N M_N^{1/2})$. Then,

$$\mathbb{E}^*(\zeta_N) = P^*(||\tilde{X}_{\Phi,B,N}\widehat{\gamma} - \tilde{X}_{\Phi,B,N}\gamma^*||_2 \gtrsim \theta_N M_N^{1/2})$$
$$= P^*(||P_{\tilde{X}_{\Phi,B,N}}\tilde{X}_{\Phi,N}\eta^* + P_{\tilde{X}_{\Phi,B,N}}\epsilon||_2^2 \gtrsim \theta_N^2 M_N)$$
$$\leq P^*(||P_{\tilde{X}_{\Phi,B,N}}\tilde{X}_{\Phi,N}\eta^*||_2^2 + ||P_{\tilde{X}_{\Phi,B,N}}\epsilon||_2^2 \gtrsim \theta_N^2 M_N),$$

where $P_{\tilde{X}_{\Phi,B,N}}$ denotes the projection matrix corresponding to the matrix $\tilde{X}_{\Phi,B,N}$. Note that

$$||P_{\tilde{X}_{\Phi,B,N}}\tilde{X}_{\Phi,N}\eta^*||_2^2 \leq \eta^{*\mathrm{T}}\tilde{X}^{\mathrm{T}}_{\Phi,N}P_{\tilde{X}_{\Phi,B,N}}\tilde{X}_{\Phi,N}\eta^* \leq ||\tilde{X}_{\Phi,N}\eta^*||_2^2.$$

We then refer to equation (19) to see that $E_{\mathcal{S}}E_X||\tilde{X}_{\Phi,N}\eta^*||_2^2 \lesssim N\kappa_N M_N^{-\xi/(\xi+1)} \prec M_N\theta_N^2$. The above two facts together conclude that

$$\mathbb{E}_{\mathcal{S}}\mathbb{E}_X[||P_{\tilde{X}_{\Phi,B,N}}\tilde{X}_{\Phi,N}\eta^*||_2^2] \lesssim N\kappa_N M_N^{-\xi/(\xi+1)} \prec M_N\theta_N^2.$$

$\mathbb{E}^*(\zeta_N) \lesssim P^*(||P_{\tilde{X}_{\Phi,B,N}}\epsilon||_2^2 \gtrsim \theta_N^2 M_N) = P^*(\epsilon^{\mathrm{T}}P_{\tilde{X}_{\Phi,B,N}}\epsilon \gtrsim \theta_N^2 M_N)$.

Note that under P^*, $\epsilon \sim N(0, \Phi\Phi^{\mathrm{T}})$, and, $e_{max}(\Phi\Phi^{\mathrm{T}}) \asymp 1$ (by Lemma 1). Also note that Lemma 1 of Laurent & Massart (2000) can be simplified to write $P^*(\chi^2_{p^*} > x) \leq \exp(-x/4)$, for $x \geq 8p^*$. Further, $\epsilon^{\mathrm{T}}P_{\tilde{X}_{\Phi,B,N}}\epsilon$ follows a χ^2 distribution with degree of freedom less than equal to $H_N\tilde{P} \prec M_N\theta_N^2 = M_N^{1/(1+\xi)}$. Using all the above facts, we conclude that $E^*(\zeta_N) \lesssim \exp(-M_N\theta_N^2)$.

Next, for $\gamma \in \mathcal{B}_N^c$, we show that $\mathbb{E}_{\mathcal{S}}\mathbb{E}_{\mathcal{X}}||\tilde{X}_{\Phi,B,N}\gamma - \tilde{X}_{\Phi,B,N}\gamma^*||_2^2 \gtrsim M_N\theta_N^2$. To see this, note that

$$\mathbb{E}_{\mathcal{S}}E_{\mathcal{X}}||\tilde{X}_{\Phi,B,N}\gamma - \tilde{X}_{\Phi,B,N}\gamma^*||_2^2 = \mathbb{E}_{\mathcal{S}}\mathbb{E}_{\mathcal{X}}\left[(\gamma-\gamma^*)^{\mathrm{T}}\tilde{X}^{\mathrm{T}}_{\Phi,B,N}\tilde{X}_{\Phi,B,N}(\gamma-\gamma^*)\right]$$
$$\asymp \kappa_N\mathbb{E}_{\mathcal{S}}\mathbb{E}_{\mathcal{X}}\left[(\gamma-\gamma^*)^{\mathrm{T}}B^{\mathrm{T}}\tilde{X}^{\mathrm{T}}_N\tilde{X}_N B(\gamma-\gamma^*)\right] \asymp N\kappa_N||\gamma-\gamma^*||_2^2/H_N \gtrsim M_N\theta_N^2,$$

where the second line follows using similar calculations leading to equation (18).

Now, using the fact that $||\tilde{X}_{\Phi,B,N}\widehat{\gamma} - \tilde{X}_{\Phi,B,N}\gamma||_2 \geq -||\tilde{X}_{\Phi,B,N}\widehat{\gamma} - \tilde{X}_{\Phi,B,N}\gamma^*||_2 + ||\tilde{X}_{\Phi,B,N}\gamma - \tilde{X}_{\Phi,B,N}\gamma^*||_2$, we obtain

$$\begin{aligned}\mathbb{E}_\gamma(1-\zeta_N) &= P_\gamma(||\tilde{X}_{\Phi,B,N}\widehat{\gamma} - \tilde{X}_{\Phi,B,N}\gamma^*||_2 \lesssim \theta_N M_N^{1/2}) \\ &= P_\gamma(||\tilde{X}_{\Phi,B,N}\widehat{\gamma} - \tilde{X}_{\Phi,B,N}\gamma||_2 \gtrsim \theta_N M_N^{1/2}) \\ &\leq P_\gamma(||P_{\tilde{X}_{\Phi,B,N}}\epsilon||_2^2 \gtrsim \theta_N^2 M_N) \lesssim \exp(-M_N\theta_N^2),\end{aligned}$$

where the last inequality follows from simplifying the conclusion for Lemma 1 of Laurent & Massart (2000) (as is done before) and the fact that under P_γ, $\epsilon \sim N(0, I)$. □

7.1. Proof of Lemma 1

Proof. Note that,

$$\begin{aligned}||w - w^*||_2 \leq ||w - \tilde{w}^* + \tilde{w}^* - w^*||_2 \leq ||w - \tilde{w}^*||_2 + ||\tilde{w}^* - w^*||_2 = ||w - \tilde{w}^*||_2 + ||\eta^*||_2 \\ \lesssim ||w - \tilde{w}^*||_2 + P^{1/2}H_N^{-\xi} \asymp ||\gamma - \gamma^*||_2 H_N^{-1/2} + P^{1/2}H_N^{-\xi} \\ \asymp ||\gamma - \gamma^*||_2 H_N^{-1/2} + P^{1/2}M_N^{-\xi/(2\xi+2)},\end{aligned}$$

where $\tilde{w}^*(s) = (\sum_{h=1}^{H_N} B_{1h}(s)\gamma_{1h}^*, \ldots, \sum_{h=1}^{H_N} B_{\tilde{P}h}(s)\gamma_{\tilde{P}h}^*)^{\mathrm{T}}$, and the first inequality in the second line follows from the property of B-splines (Huang et al., 2004). The second expression in the second line follows from Lemma A.1 of Huang et al. (2004). Using the fact that $\tilde{P}^{1/2}M_N^{-\xi/(2\xi+2)} = O(\theta_N)$, we have $\left\{w : ||w - w^*||_2 \geq \tilde{C}\theta_N\right\} \subset \left\{\gamma : ||\gamma - \gamma^*||_2 H_N^{-1/2} \geq C_{2w}\theta_N\right\}$, for some constant $C_{2w} > 0$.

Denote $\mathcal{B}_N = \left\{\gamma : ||\gamma - \gamma^*||_2 H_N^{-1/2} \leq C_{2w}\theta_N\right\}$. To prove the theorem, it is enough to establish

$$\mathbb{E}^*\Pi(||\gamma - \gamma^*||_2 H_N^{-1/2} \geq C_{2w}\theta_N | y_{\Phi,N}, \tilde{X}_{\Phi,N}) \to 0, \text{ as } N \to \infty, \tag{20}$$

Note that,

$$\begin{aligned}\mathbb{E}^*[\Pi(\mathcal{B}_N^c \mid y_{\Phi,N}, \tilde{X}_{\Phi,N})] &\leq \mathbb{E}^*\zeta_N + \mathbb{E}^*[\Pi(\mathcal{B}_N^c \mid y_{\Phi,N}, \tilde{X}_{\Phi,N})(1-\zeta_N)1_{y_N \in \mathcal{A}_N^c}] + P^*(\mathcal{A}_N) \\ &= \mathbb{E}^*[\zeta_N] + \mathbb{E}^*\left[1_{y_N \in \mathcal{A}_N^c} \frac{\left\{(1-\zeta_N)\int_{\mathcal{B}_N^c}\{f(y_{\Phi,N}|\gamma)/f^*(y_{\Phi,N}|\gamma^*)\}\pi_N(\gamma)d\gamma\right\}}{\left\{\int\{f(y_{\Phi,N}|\gamma)/f^*(y_{\Phi,N}|\gamma^*)\}\pi_N(\gamma)d\gamma\right\}}\right] + P^*(\mathcal{A}_N),\end{aligned} \tag{21}$$

where $\mathcal{A}_N$ is a set defined in the statement of Lemma 2 and ζ_N can be regarded as a sequence of random variables as defined in Lemma 3. By Lemma 2, $P^*(\mathcal{A}_N) \to 0$, as $N, M_N \to \infty$. Also, by Lemma 3, $\mathbb{E}^*\zeta_N \to 0$, as $N, M_N \to \infty$. To show (20), it remains to prove that

$$\frac{\mathbb{E}^*\left[1_{y_N \in \mathcal{A}_N^c}\int_{\mathcal{B}_N^c}\{f(y_{\Phi,N}|\gamma)/f^*(y_{\Phi,N}|\gamma^*)\}\pi_N(\gamma)d\gamma\right]}{\left[\int\{f(y_{\Phi,N}|\gamma)/f^*(y_{\Phi,N}|\gamma^*)\}\pi_N(\gamma)d\gamma\right]} \to 0 \text{ as } N, M_N \to \infty.$$

To this end, we have

$$\begin{aligned}\mathbb{E}^*\left[1_{y_N \in \mathcal{A}_N^c}\int_{\mathcal{B}_N^c}\{f(y_{\Phi,N}|\gamma)/f^*(y_{\Phi,N}|\gamma^*)\}\pi_N(\gamma)d\gamma\right] &\leq \sup_{\gamma \in \mathcal{B}_n^c}\mathbb{E}_\gamma(1-\zeta_N)\Pi(\mathcal{B}_N^c) \\ &\leq \exp(-C_{2w}M_N\theta_N^2),\end{aligned}$$

where $\Pi(\mathcal{B}_N^c)$ is the prior probability of the set $\mathcal{B}_N^C$. The denominator $\int\{f(y_{\Phi,N}|\gamma)/f^*(y_{\Phi,N}|\gamma^*)\}\pi(\gamma)d\gamma \geq \exp(-C_1 M_N\theta_N^2)$ on $\mathcal{A}_N$, where C_1 is chosen so that $C_1 < C_{2w}$. Thus, $\mathbb{E}^*\Pi(\mathcal{B}_N^c \mid y_{\Phi,N}, \tilde{X}_{\Phi,N})1_{y_N\in\mathcal{A}_N^c} \leq \exp(-(C_{2w}-C_1)M_N\theta_N^2) \to 0$, as $N, M_N \to \infty$. □

7.2. *Proof of Theorem 2*

Proof. For densities f_u and f^*, we have

$$\begin{aligned} h(f_u, f^*) &= 1 - \exp\left\{-\left(\sum_{j=1}^{\tilde{P}} \tilde{x}_j(s_0)w_j(s_0) - \sum_{j=1}^{\tilde{P}} \tilde{x}_j(s_0)w_j^*(s_0)\right)^2 /8\right\} \\ &\leq 1 - \exp\left\{-\tilde{P}\sum_{j=1}^{\tilde{P}} \left(w_j(s_0) - w_j^*(s_0)\right)^2 /8\right\} \\ &\leq 1 - \exp\left\{-\tilde{P}||w(s_0) - w^*(s_0)||_2^2/8\right\} \end{aligned}$$

Then, $\mathbb{E}_{\mathcal{S}}[h(f_u, f^*)] \leq 1 - \exp\left(-\tilde{P}||w - w^*||_2^2/8\right)$, by Jensen's inequality. Further,

$$\mathbb{E}^*\mathbb{E}\mathbb{E}_{\mathcal{S}}[h(f_u, f^*)|\tilde{X}_{\Phi,N}, y_{\Phi,N}] = \left\{1 - \exp\left(-\tilde{P}\tilde{C}^2\theta_N^2/8\right)\right\} + 2\Pi_N(||w - w^*||_2 \geq \tilde{C}\theta_N),$$

which implies

$$\mathbb{E}^*\mathbb{E}\mathbb{E}_{\mathcal{S}}[h(f_u, f^*)] \leq \left\{1 - \exp\left(-\tilde{P}\tilde{C}^2\theta_N^2/8\right)\right\} + 2\mathbb{E}^*\Pi_N(||w - w^*||_2 \geq \tilde{C}\theta_N) \to 0$$

as $N, M_N \to \infty$, where the last expression followed by the conclusion of Theorem 1 and the fact that $\theta_N \to 0$ as $N, M_N \to \infty$. □

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