# A Botnet Detection Game

Braden Soper and John Musacchio

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**Abstract**

Botnets continue to constitute a major security threat to users of the internet. We examine a novel security game between the operator of a botnet and the legitimate users of the compromised network. The more a btotmaster utilizes his botnet, the more likely it is he will be detected by the legitimate users of the network. Thus he must balance stealth and aggression in his strategic utilization of his botnet. The legitimate users of the network then must decide how vigilant they will be in trying to detect the presence of the botnet infection.

# Contents

# 1 Introduction

As one of the major security threats to users of the internet, botnets exemplify the difficulties of network security. They are highly distributed, interconnected and complex. Furthermore a single botnet can contain thousands of computers, making scores of legitimate computer users unwitting participants in cyber-criminal activities. Though the research community has taken an interest in the botnet phenomena, theoretical models of botnets are nascent.

Game theory has become an important modeling tool for network security [3, 16] and we believe botnets are no exception. Standard security games deal with the traditional *attacker-defender* dynamic. When modeling a botnet security game the strategic attacker is the cyber-criminal controlling the botnet, often called the *bot herder* or *bot master*. However, modeling the defender is not so clear cut. Typically the defender is thought to be the intended target of an attack. In the case of botnets this might be the security administrator of a server under a DDoS attack or a network spam filter being bombarded by a spambot. A missing component of such models are the legitimate computer users whose compromised computers, often called *zombies*, make up the botnet. These individuals are the first line of defense against botnets and understanding the interaction between them and the bot master is crucial to understanding the botnet threat.

We propose a novel security game which explicitly considers the bot master as a strategic agent pitted against the legitimate users of a computer network targeted to become a botnet. In our game the bot master must consider the tradeoffs between stealth and aggressiveness in his utilization of his botnet. The legitimate users of the network (*agents*) act as intrusion detection systems and must consider the tradeoffs between the costs associated with false alarms (false positives) and the losses associated with missed detections (false negatives).

One difficulty in dealing with this type of security threat is the distributed, interconnected nature of the agents and their host computers. We propose to extend the Local Mean Field (LMF) model in [11] to address this difficulty. In [10] the author suggests a LMF approach to modeling botnets, but the bot master is not explicitly considered as a strategic agent in their game. Furthermore the agents in [10] and [11] are not trying to detect the presence of an infection, but are deciding whether or not to invest in security.

Previous work on network security games have considered similar issues. In [17] economic models are considered to study the incentives of ad-networks and ISPs to invest in detecting botnets. Interconnected agents, network externalities and security investments have been considered in [9] [14] [13] [4] [5] [8] and [11]. Botnet dynamics have been considered in [6], while botnet economics are considered in [15] and [12].

## 2 Two-person Botnet Detection Game

We consider a security game between a bot master (attacker) and the owner of a single targeted computer (defender). The bot master tries to gain control of the computer directly with some probability of success $p$. If the infection is successful then the bot master gains control of the target computer and can use it for his own nefarious purposes (spam, DDoS Attacks, click-fraud, etc.). If the botmaster is too aggressive in his use of the compromised computer, then it is more likely that the defender will detect the infection. For example, if the bot master is continually using the compromised computer to send large volumes of spam, there may be a noticeable slow down in performance in the defender's computer due to excessive bandwidth consumption as a result of the spamming. The defender may then decide it is time to patch/replace his computer, thus ridding himself of the infection. We begin by modeling this simple two-player game and then extend the game to large networks of targeted computers.

To motivate our game we consider the fact that cyber-criminals are able to compromise computers by exploiting flaws in the software and hardware of networked systems. A game theoretic model of software/hardware manufacturers and their incentives to invest in reducing software system failures is presented in [7] . The authors classify software failures into two categories: security failures (failures caused by malicious and unauthorized access to a user's system) and reliability failures (those failures which are not security failures). Two observations on which the authors base their game are 1) the source of security failures and reliability failures is the same (software bugs), and 2) it is too costly for users to distinguish between the two types of failures. We incorporate these observations into our game by modeling security failures, reliability failures and a user's inability to distinguish between them.

We assume the bot master tries to infect the targeted computer as aggressively as possible and has an overall probability of success $p$. Our focus is not on the strategy of the bot master's initial infection attempt, but on his strategic behavior in utilizing a compromised computer once he "owns" it. The bot master infects computers in order to illegitimately utilize available computational resources, i.e. CPU time, RAM , bandwidth, electrical power, etc. Specifically we wish to model how *aggressive* the bot master should be in utilizing these resources. In what follows we will not model a particular resource, instead we model a general measurable resource $R$ taking values on $\mathbb{R}^+$. The strategic variable for the bot master is a measure of his aggressiveness in utilizing the resource $R$. We call this value $A$ and assume it is chosen from a strategy space $\mathcal{A}$.

We can think of $A$ as a proxy for how much of the infected computer's resources the attacker decides to use. In fact we assume there exists a one-to-one mapping $b : \mathcal{A} \to \mathbb{R}^+$ such that $b(A)$ is the directly observable amount of resource $R$ the bot master uses. In the context of [7] we interpret $b(A)$ as the observable security failures associated with the resource $R$ of an infected computer. We model the observable reliability failures associated with resource $R$ as a random variable $S$ with support $\mathbb{R}^+$. We denote the cumulative distribution function of $S$ by $F_S(x)$ and the probability density function by $f_S(x)$. We will assume that $F_S(x)$ is strictly increasing and smooth and that $f_S(x)$ is differentiable and strictly positive for all $x > 0$.

Suppose that the typical resource usage directly attributable to the agent is $R_0$. Let $W \sim$ Bernoulli$(p)$ where $W = 1$ if and only if the direct infection of the bot master is successful and $W = 0$ otherwise. We assume $W$ and $S$ are independent. Then the total resource usage $R$ is a random variable which can be modeled as

$$R = R_0 + b(A)W + S.$$

The computer owner, or agent, is thought to be a typical user of a computer connected to the internet. Aware that there are potential security threats the agent must decide how vigilant he will be in detecting such threats. As in [7] we assume the agent is unable to reliably distinguish between security failures and reliability failures and thus must consider the potential costs from both false alarms (false positives) and missed detections (false negatives). Assuming the value $R_0$ is known, the agent makes a single observation $Z = R - R_0$ of his system's software failures, i.e.

$$Z = b(A)W + S.$$

Given the observation $Z$ the agent wishes to determine whether or not his computer is infected. Assuming the distributions of $S$ and $W$ are known this becomes a simple hypothesis testing problem.

$$H_0 : W = 0$$
$$H_1 : W = 1$$

Given that higher observed values of $Z$ indicate a higher likelihood of infection, we take the strategic variable of the agent to be a threshold $T$ in a strategy space $\mathcal{T} \subseteq \mathbb{R}^+$.[1] We interpret this as a measure of the agent's tolerance for software failures. If $Z \geq T$ the agent decides his computer is infected and takes appropriate measures to remediate the potential infection. If $Z < T$ then the agent takes no action. For notational simplicity we define the indicator random variable $D = \mathbb{1}_{\{Z \geq T\}}$, i.e. $D = 1$ if and only if $Z \geq T$ and $D = 0$ otherwise.

We construct a loss function for the agent by first defining the following indicator random variables.

$$X_1 = \begin{cases} 1 & \text{if False Positive} \\ 0 & \text{o.w.} \end{cases}$$

[1] More generally the agent is free to choose any decision rule mapping observations in $\mathbb{R}^+$ to the set $\{0, 1\}$. If $\mathcal{C}$ is the set of all such decision rules we assume the agent will choose a $g \in \mathcal{C}$ that minimizes his expected posterior loss, i.e. he chooses a Bayesian decision rule. Under the assumptions of our model it can be shown that the Bayes decision rule $g^* \in \mathcal{C}$ is equivalent to the above threshold classifier.

$$X_2 = \begin{cases} 1 & \text{if False Negative} \\ 0 & \text{o.w.} \end{cases}$$

$$X_3 = \begin{cases} 1 & \text{if True Positive} \\ 0 & \text{o.w.} \end{cases}$$

$$X_4 = \begin{cases} 1 & \text{if True Negative} \\ 0 & \text{o.w.} \end{cases}$$

We associate the following costs with each outcome of the detection.

$$\begin{aligned} &c_1 : \text{cost of False Positive} \\ &c_2 : \text{cost of False Negative} \\ &c_3 : \text{cost of True Positive} \\ &c_4 : \text{cost of True Negative} \end{aligned}$$

The $X_i$ all depend on the strategies chosen by the players, and in general so do the $c_i$. For $A \in \mathcal{A}$ and $T \in \mathcal{T}$ the defender's cost is defined as

$$C(A,T) = \sum_{1 \le i \le 4} c_i(A,T) X_i(A,T).$$

The relationship between the $X_i$, $W$ and $D$ can be expressed in the following way.

$$\begin{aligned} X_1 &= (1 - W)D \\ X_2 &= W(1 - D) \\ X_3 &= WD \\ X_4 &= (1 - W)(1 - D) \end{aligned}$$

Taking expectations we have

$$\begin{aligned} E[X_1] &= [1 - F_S(T)](1 - p) \\ E[X_2] &= F_S(T - b(A))p \\ E[X_3] &= [1 - F_S(T - b(A))]p \\ E[X_4] &= F_S(T)(1 - p) \end{aligned}$$

We assume the following cost functions in our model.

1. $c_1(a,t) \equiv r + k$ for constants $r \ge 0$ and $k \ge 0$

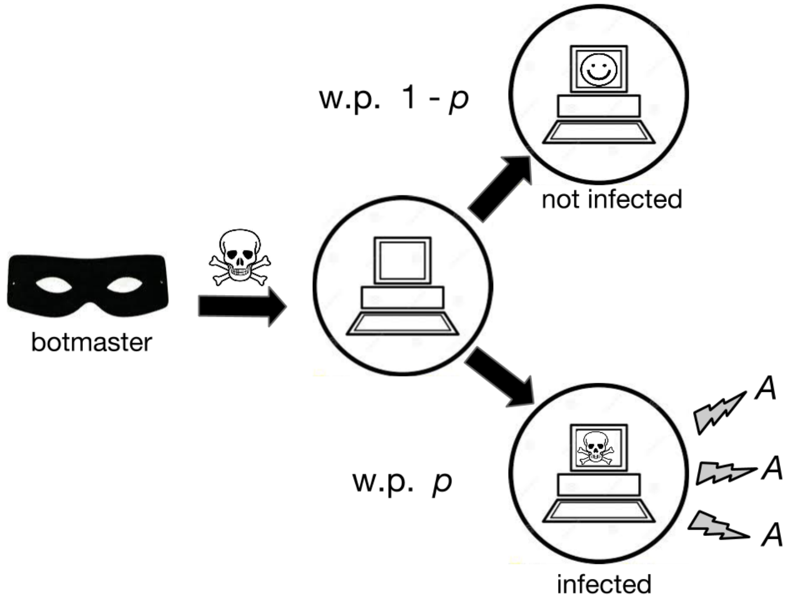



Figure 1: The botmaster attempts to infect a computer with probability of success $p$. If successful the botmaster uses the compromised computer to launch attacks of strength $A$.

2. $c_2(a,t) \equiv v(a) : \mathcal{A} \to \mathbb{R}^+$ is differentiable and non-decreasing with $v(0) = 0$.
3. $c_3(a,t) \equiv k$
4. $c_4(a,t) \equiv 0$

The function $v(\cdot)$ represents the cost associated with future lost resources when the detection is missed. The value $k$ is the fixed cost associated with remediating a potential infection such as reinstalling an operating system, updating software or purchasing a new computer. Notice this cost is incurred whenever the infection is present, regardless of the agent's decision. Cost $r$ is the additional cost associated with a false alarm not incurred during a true detection. Notice $c_4 \equiv 0$ since the agent incurs no cost if he correctly identifies that his system is not infected. The expected cost of the defender is now

$$E[C(A,T)] = (r+k)[1 - F_S(T)](1-p) + [k + (v(A) - k)F_S(T - b(A))]p$$

Notice that a successful attack is the same as a false negative and has indicator random variable $X_1$. Define the function $g : \mathcal{A} \to \mathbb{R}^+$ to be the utility gained from a successful attack given an aggressiveness $A$. We assume $g(A)$ is twice differentiable with $\frac{dg}{dA} > 0, \frac{d^2g}{dA^2} \leq 0$ and $g(0) = 0$. We define the attacker utility as

$$U(A,T) = g(A)X_1(A,T).$$

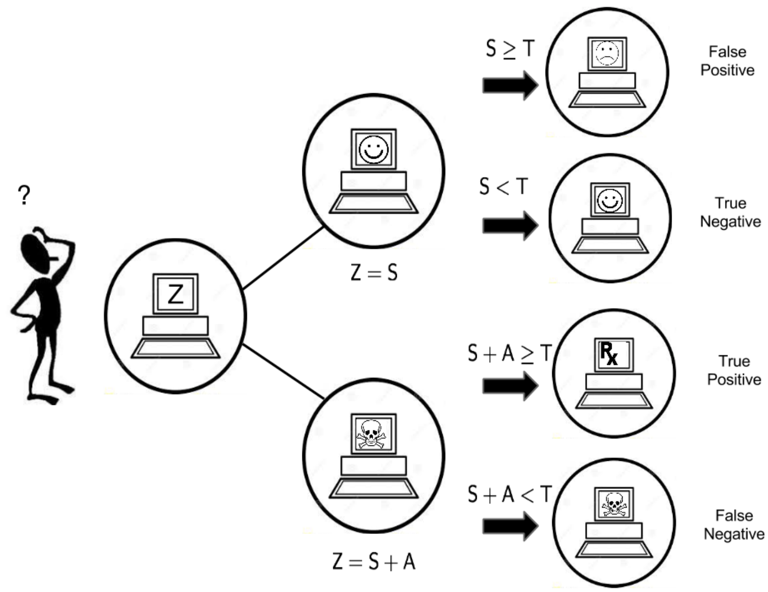



Figure 2: An agent makes an observation $Z$ of his computer, not knowing its true state (infected/not infected). After comparing the observation $Z$ to his chosen tolerance $T$ he decides whether or not to take action. The four possible outcomes are depicted. The agent incurs a cost for false negatives and false positives while no cost is incurred for true positives and true negatives.

The expected utility is then

$$E[U(A,T)] = g(A)F_S(T - b(A))p$$

Notice the relationship between the functions $b(A), v(A)$ and $g(A)$. The function $b(A)$ is a direct measurement of stolen resources while $v(A)$ and $g(A)$ are the valuations of the stolen resources by the defender and attacker respectively. Since $b(A)$ is one-to-one we may take the strategy of the attacker to be $\bar{A} = b(A)$ with strategy space $\bar{\mathcal{A}} = \mathbb{R}^+$. Thus without loss of generality and a recycling of notation we assume $b(A) \equiv A$ and $\mathcal{A} = \mathbb{R}^+$. We then have the expected cost/utility functions as follows.

$$E[C(A,T)] = (r+k)[1 - F_S(T)](1-p) + [k + (v(A) - k)F_S(T - A)]p$$
$$E[U(A,T)] = g(A)F_S(T - A)p$$

We can then define the best response correspondences as follows.

$$\sigma_1(A) = \arg\min_T E[C(A,T)]$$
$$\sigma_2(T) = \arg\max_A E[U(A,T)]$$

The following proposition characterizes the pure Nash equilibrium in the two player game for the case $S \sim \text{gamma}(\alpha, \beta)$ with $\alpha \geq 1$. We state the proposition without proof as the arguments needed are developed in the sequel.

**Proposition 1.** *Suppose $S \sim gamma(\alpha, \beta)$ with $\alpha \geq 1$. Then there exists a unique pure Nash equilibrium $(A^*, T^*)$ in the two-player botnet game with $T^* > A^* > v^{-1}(k)$. For the case $\alpha = 1$ the solution is given by the following equations.*

$$A^* = \frac{1}{\beta} \log\left(\frac{r+k}{v(A^*)-k}\frac{1-p}{p}\right)$$
$$T^* = A^* + \frac{1}{\beta}\log\left[1 + \frac{g(A^*)}{g'(A^*)}\right]$$

*For the case $\alpha > 1$ the solution is given by the following equations.*

$$T^* = A^*\left(1 - \left[e^{-\beta A^*}\frac{r+k}{v(A^*)-k}\frac{1-p}{p}\right]^{\frac{1}{\alpha-1}}\right)^{-1}$$
$$\frac{g(A^*)}{g'(A^*)} = \frac{\int_0^{T^*-A^*} x^{\alpha-1}e^{-\beta x}dx}{(T^*-A^*)^{\alpha-1}e^{-\beta(T^*-A^*)}}$$

# 3 A Large Population Botnet Game

We now extend the botnet detection game introduced above to a game with a large number of agents in a network. In [11] network externalities in a security investment game between a large number of interconnected agents are studied. Their so called Local Mean Field model focuses on the asymptotic properties of the game in the limit of a large number of agents via the objective method [2]. In [10] the auhor discusses applying the model to study botnets, but the model presented focused on agent incentives to invest in security and did not explicitly incorporate the bot master into the model. We wish to extend the LMF model introduced in [11], explicitly modeling the bot master as a strategic agent. Following [11] we consider a sequence of Erdos-Renyi random graphs $G(n, \lambda/n)$ and look for solutions on the limiting graph as $n \to \infty$. Because Erdos-Renyi graphs converge to a Galton-Watson Poisson Branching Process, denoted by $T(\lambda)$, in the sense of local weak convergence [2], we restrict our analysis to $T(\lambda)$. The convergence results which justify this step can be found in section **??**. For details on the objective method, local weak convergence and random distributional equations readers are referred to [2] and [1].

With each computer on the network we associate a unique agent $a_i$ with $i = 1, 2, 3, \ldots$. For simplicity we assume all agents are homogeneous. In particular they have the same cost functions and each agent is equally likely to occupy any place in the network. A root agent is chosen uniformly at random from all agents in the network. This agent will be crucial in our equilibrium analysis. We designate the root agent by $a_\emptyset$. As in the two person game above, we assume agents use a threshold decision rule to detect infections. We let $A \in \mathcal{A}$ be the aggressiveness of the botmaster, $T_i \in \mathcal{T}$ the threshold of agent $i$, and $Z_i \in \mathcal{S}$ the observation made by agent $i$. If $Z_i \geq T_i$ then the agent concludes his system has been compromised and takes measures to remediate the problem. If $Z_i < T_i$ then no infection is detected. In our model we assume that if the agent detects the infection, then he stops the infection and does not pass it on to his neighbors. Let $S_i$ denote the random system noise observed by agent $a_i$. We assume the $S_i$ are i.i.d. for all $i = 1, 2, 3, ...$, again denoting the cumulative distribution function by $F_S(x)$ and the probability density function by $f_S(x)$ with support $\mathcal{S}$.

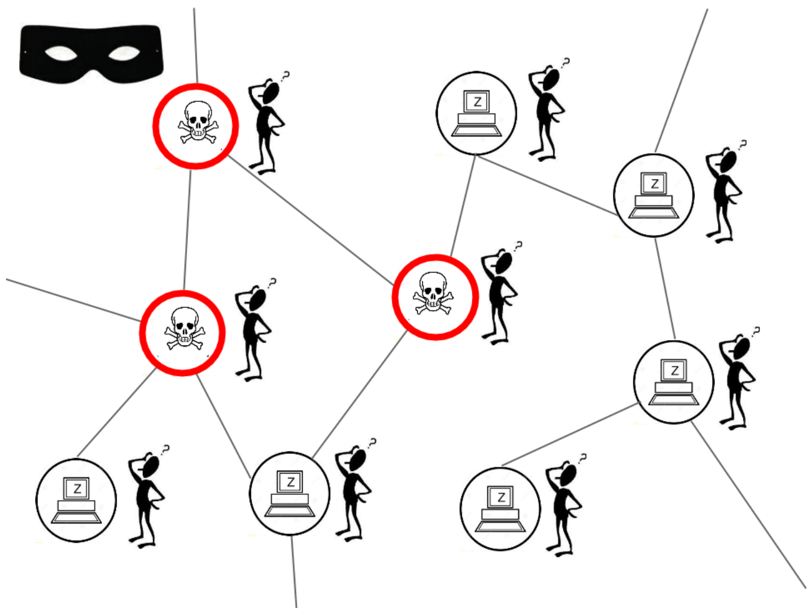


Figure 3: When agents are connected in a network, the strategic behavior of each agent affects the strategic behavior of all other agents as well as that of the botmaster.

Our model on $T(\lambda)$ is characterized by the following stochastic processes following [11]. Let the random variables $\chi_i \overset{i.i.d.}{\sim}$ Bernoulli($p$) indicate a direct infection of agent $i$ by the bot master, and let the random variables $B_{kj} \overset{i.i.d.}{\sim}$ Bernoulli($q$) indicate contagion from agent $k$ to agent $j$ for all $k \neq j \in T(\lambda)$. For each $i \in T(\lambda)$ let $W_i$ be the indicator random variable indicating that infection reaches agent $i$ either from a direct descendant in $T(\lambda)$ or directly from the bot master. Let $D_i$ be the indicator random variable indicating whether agent $i$ detects such an infection. We can then define detection outcome indicator random variables as follows.

$$\begin{aligned}
X_i &= (1 - W_i)D_i \\
Y_i &= W_i(1 - D_i) \\
\tilde{X}_i &= W_i D_i \\
\tilde{Y}_i &= (1 - W_i)(1 - D_i)
\end{aligned}$$

Thus $X_i$ indicates a false positive, $Y_i$ a false negative, $\tilde{X}_i$ a true positive and $\tilde{Y}_i$ a true negative by agent $i \in T(\lambda)$ when infected from a direct descendant in $T(\lambda)$ or directly from the bot master. The equations for $W_i$ and $D_i$ are

$$\begin{aligned}
W_i &= 1 - (1 - \chi_i) \prod_{k \to i} (1 - B_{ki} W_k (1 - D_k)), \\
D_i &= \mathbb{1}_{\{S_i + W_i A \geq T_i\}}
\end{aligned}$$

Here $k \to i$ denotes that agent $k$ is a direct descendant of agent $i$ in the rooted tree. The equation for the observation $Z_i$ is then

$$Z_i = S_i + W_i A.$$

The introduction of the processes $W_i$ and $D_i$ are done to take advantage of the structure of $T(\lambda)$. It is important to note that $W_i$ and $W_j$ are independent of one another when $a_i$ and $a_j$ are the same distance away from the root node. In fact because $W_i$ and $D_i$ depend only on the children of $a_i$, the processes $\{W_i\}_{i \in T(\lambda)}$, $\{X_i\}_{i \in T(\lambda)}$, $\{Y_i\}_{i \in T(\lambda)}$, $\{\tilde{X}_i\}_{i \in T(\lambda)}$ and $\{\tilde{Y}_i\}_{i \in T(\lambda)}$ are Recursive Tree Processes. As in [11] it is this structure that makes the model tractable.

Because the cost function is the same for all agents, and each agent's *a priori* knowledge of the network is the same, we focus our analysis on finding a symmetric equilibrium among the agents. In particular we seek a Nash equilibrium in which all agents choose the same strategy $T$. Clearly there may be equilibrium solutions in which players select different strategies, but for tractability we focus on symmetric, mutual best responses. Because the root is chosen uniformly at random we expect there to be an invariant process [1]. The fundamental recursive distributional equations which define the invariant process on $T(\lambda)$ are as follows.

$$W \stackrel{d}{=} 1 - (1 - \chi) \prod_{k=1}^{N} (1 - B_k Y_k) \tag{1}$$

$$D \stackrel{d}{=} \mathbb{1}_{\{T \leq S + WA\}} \tag{2}$$

$$Y \stackrel{d}{=} W(1 - D) \tag{3}$$

The random variable $\chi \sim \text{Bernoulli}(p)$, $S \sim \text{Gamma}(\alpha, \beta)$, $B_k \overset{iid}{\sim} \text{Bernoulli}(q)$ and $N \sim \text{Poisson}(\lambda)$ are random variables independent of everything in the model. The random variables $Y$ and $Y_k$ $k = 1, 2, ..., N$ are i.i.d copies satisfying (3). Notice that the random variable $Y$ plays a special role in these equations because the distribution of $W$ depends on the distribution of $Y$. Provided solutions exist for equations (1)-(3) we can also define the other detection indicator random variables.

$$\begin{aligned} X &\overset{d}{=} (1 - W)D \\ \tilde{X} &\overset{d}{=} WD \\ \tilde{Y} &\overset{d}{=} (1 - W)(1 - D) \end{aligned}$$

We now find a solution to (3). This result is analogous to Prop. 2 in [11].

**Proposition 2.** *Let $S \sim F_S(\cdot)$, $A, T \in \mathcal{S}$ with $T \geq A$, $0 < p \leq 1$ and $0 < q \leq 1$. Then the RDE for $Y$ has a unique solution: $\mathbb{P}(Y = 1) = 1 - \mathbb{P}(Y = 0) = h$, where $h = h(A, T, p, q, \lambda, F_S(\cdot))$ is the unique solution in $[0, 1]$ of the fixed point equation*

$$h = F_S(T - A)[1 - (1 - p)e^{-\lambda q h}]. \tag{4}$$

*Proof.* Let $h = \mathbb{P}(Y = 1)$. Then

$$\begin{aligned} h &= \mathbb{P}(Y = 1|W = 1)P(W = 1) + \mathbb{P}(Y = 1|W = 0)P(W = 0) \\ &= \mathbb{P}(Y = 1|W = 1)P(W = 1). \end{aligned}$$

Conditioned on $W = 1$ the distributional equation reduces to $Y \overset{d}{=} \mathbb{1}_{\{T > S + A\}}$, giving us

$$\mathbb{P}(Y = 1|W = 1) = \mathbb{P}(T > S + A) = F_S(T - A).$$

The distribution for $W$ satisfies

$$\begin{aligned} \mathbb{P}(W = 0) &= \mathbb{P}((1 - \chi)\prod_{k=1}^{N}(1 - B_k Y_k) = 1) \\ &= (1 - p)\sum_{n=0}^{\infty}(1 - \mathbb{P}(B_k Y_k = 1))^n\, \mathbb{P}(N = n) \\ &= (1 - p)\sum_{n=0}^{\infty}(1 - qh)^n \frac{\lambda^n e^{-\lambda}}{n!} \\ &= (1 - p)e^{-\lambda q h}. \end{aligned}$$

Thus $\mathbb{P}(W = 1) = 1 - (1 - p)e^{-\lambda q h}$, giving us the following fixed point equation.

$$h = F_S(T-A)\left[1-(1-p)e^{-\lambda q h}\right]$$

Let $f(x,T,A) = F_S(T-A)[1-(1-p)e^{-\lambda q x}]$. Then $f$ is continuous, increasing and concave in $x$. Since $f(0,\gamma,T,A) > 0$ and $f(1,\gamma,T,A) \leq 1$ there must be a unique fixed point of $f$ which depends on $A, T, p, q, \lambda, F_S(\cdot)$. $\square$

It is important to keep in mind that $h$ depends on all parameters and choice variables of the model. In particular we will be interested in $h(A,T)$. We will often suppress this dependence in the notation for the sake of brevity. With Prop. 2 the distributions of the remaining detection indicator random variables can be obtained.

**Corollary 1.** *Let $S \sim F_S(\cdot)$, $A, T \in \mathcal{S}$ with $T \geq A$, $0 < p \leq 1$ and $0 < q \leq 1$. Then the RDEs for $X, \tilde{X}$ and $\tilde{Y}$ have unique solutions which depend on the distribution of $Y$. In particular if $E[Y] = h(A,T)$ then*

$$\begin{aligned} E[X] &= [1-F_S(T)](1-p)e^{-\lambda q h(A,T)}, \\ E[\tilde{X}] &= [1-F_S(T-A)][1-(1-p)e^{-\lambda q h(A,T)}], \\ E[\tilde{Y}] &= F_S(T)(1-p)e^{-\lambda q h(A,T)}. \end{aligned}$$

Before solving the bot master's and agents' optimization problems we will need some results on the function $h(A,T)$. By the Implicit Function Theorem $h$ is differentiable in $A$ and $T$ provided that $f_S(T-A) \neq 0$. A direct computation of the derivative of $h$ with respect to $A$ gives

$$\frac{\partial h}{\partial A} = -f_S(T-A)[1-(1-p)e^{-\lambda q h}] + F_S(T-A)\left[\lambda q \frac{\partial h}{\partial A}(1-p)e^{-\lambda q h}\right].$$

Solving for $\frac{\partial h}{\partial A}$ we have

$$\frac{\partial h}{\partial A} = -\frac{f_S(T-A)[1-(1-p)e^{-\lambda q h}]}{1-F_S(T-A)[\lambda q(1-p)e^{-\lambda q h}]}. \tag{5}$$

The following lemma guarantees the boundedness of the derivatives of $h$. It will also be useful in further analysis.

**Lemma 1.** *Define $\theta(A,T) = 1 - \lambda q F_S(T-A)(1-p)e^{-\lambda q h}$ where $h$ is defined as in Prop. 2. For any $\lambda q > 0$, $0 \leq p < 1$ and $T \geq A \geq 0$, we have $0 < \theta(A,T) < 1$.*

*Proof.* See Appendix A.1. $\square$

We now have that $\frac{\partial h}{\partial A} \leq 0$. Notice that $\frac{\partial h}{\partial T} = -\frac{\partial h}{\partial A}$, thus the same analysis above gives us $\frac{\partial h}{\partial T} \geq 0$. Furthermore we see that $h(A,T)$ may not be differentiable at $A = T$ since $\frac{\partial h}{\partial A} \equiv 0$ for all $A > T$ but $\frac{\partial h}{\partial A} > 0$ for all $A < T$. In particular if $f_S(x) > 0$ at $x = 0$ we will have $\lim_{T \downarrow A} \frac{\partial h}{\partial A} > 0 = \lim_{T \uparrow A} \frac{\partial h}{\partial A}$. Other important properties of $h(A,T)$ that will be useful are that the dependence on $A$ and $T$ appear only in the form $T - A$ as arguments in $F_S(\cdot)$. Thus $h(A,T) = 0$ for all $(A,T)$ pair with $A \geq T$. Furthermore for any finite $A$ the limiting value of $h(A,T)$ as $T \to \infty$ is the same since for fixed finite $A$ we have

$$\begin{aligned}\lim_{T\to\infty} h(A,T) &= \lim_{T\to\infty} F_S(T-A)[1-(1-p)e^{-\lambda q h(A,T)}] \\ &= \lim_{T\to\infty} [1-(1-p)e^{-\lambda q h(A,T)}] \\ &= 1-(1-p)e^{-\lambda q \lim_{T\to\infty} h(A,T)}\end{aligned}$$

We will denote this limiting value by $h_\infty = \lim_{T\to\infty} h(A,T)$ and observe that it satisfies the fixed point equation

$$h_\infty = 1-(1-p)e^{-\lambda q h_\infty}. \tag{6}$$

Note that in this limit we recover the probability of infection in [11].

## 3.1 Utility and Cost Functions

### 3.1.1 Agent Cost

The agents in our game are statistically homogeneous. For this reason we are interested in finding a pure, symmetric, mutual strategy $T^*$ that all agents in the network will play in response to a strategy $A$ such that no individual agent has an incentive to unilaterally deviate. As we mentioned before, this is clearly a restrictive class of Nash equilibria and many non-symmetric equilibria may exist. But motivated by the homogeneity of the agents and a desire for tractability, we feel it is a reasonable class of equilibria to study. Future work will entail studying non-symmetric, pure equilibria, mixed equilibria as well as equilibria among heterogeneous agents.

We assume each agent is acting independently and has no knowledge of other agents other than their similarity in cost function. We investigate what happens when a single agent deviates from a population threshold $T$ that all other agents in the network are playing. Because we are concerned with the behavior of a "typical" agent, and the root of each graph is chosen uniformly at random, we consider the root agent to be our "typical" agent. Thus for a fixed strategy $A$ and a fixed network strategy $T$ we can define a deviant node's strategy $T_\varnothing$, expected cost function $C_\varnothing(A,T,T_\varnothing)$ and best response correspondence $\sigma_\varnothing(A,T)$. One can think of this process as a network population game within our attacker-defender game where for a fixed strategy $A$ we are looking for optimal network strategies $T^*$ such that $T^* \in \sigma_\varnothing(A,T^*)$.

From the recursive distributional equations (1)-(3) it is clear that if the root agent $a_\emptyset$ changes his threshold $T_\emptyset \neq T$, this will change his decision rule, but it will not change what he observes. In other words the probability of a false alarm will change, but the probability of infection does not change. Recall the variables $Y_i$ tell us whether or not the infection reaches agent $i$ from his children. It is not the probability of infection for agent $i$, since agent $i$ can be infected from it's parent. So if the root agent deviates from the population strategy $T$, then equations (1)-(3) are still valid for all agents in the tree except the root. For the root we need to introduce new distributional equations.

$$
\begin{aligned}
D_\emptyset &\stackrel{d}{=} \mathbb{1}_{\{T_\emptyset < S+WA\}} \\
X_\emptyset &\stackrel{d}{=} (1-W)D_\emptyset \\
Y_\emptyset &\stackrel{d}{=} W(1-D_\emptyset) \\
\tilde{X}_\emptyset &\stackrel{d}{=} WD_\emptyset \\
\tilde{Y}_\emptyset &\stackrel{d}{=} (1-W)(1-D_\emptyset)
\end{aligned}
$$

The corresponding distributions computed as in Prop. 2 and Cor. 1 are computed analogously.

**Proposition 3.** *Let $S \sim F_S(\cdot)$, $A, T, T_\emptyset \in \mathcal{S}$ with $T, T_\emptyset \geq A$, $0 < p \leq 1$ and $0 < q \leq 1$. Then the RDEs for $X_\emptyset, Y_\emptyset, \tilde{X}_\emptyset$ and $\tilde{Y}_\emptyset$ have unique solutions which depend on the distribution of $Y$. If $E[Y] = h(A,T)$ then the distributions are given by the following.*

$$
\begin{aligned}
E[X_\emptyset] &= [1 - F_S(T_\emptyset)](1-p)e^{-\lambda q h(A,T)} \\
E[Y_\emptyset] &= F_S(T_\emptyset - A)[1 - (1-p)e^{-\lambda q h(A,T)}] \\
E[\tilde{X}_\emptyset] &= [1 - F_S(T_\emptyset - A)][1 - (1-p)e^{-\lambda q h(A,T)}] \\
E[\tilde{Y}_\emptyset] &= F_S(T_\emptyset)(1-p)e^{-\lambda q h(A,T)}
\end{aligned}
$$

The root agent's cost function $C_\emptyset(A, T, T_\emptyset)$ can now be constructed. As in the two-person game we let $r, k \geq 0$ be constants and $v : \mathcal{A} \to \mathbb{R}^+$ be a differentiable, non-decreasing function with $v(0) = 0$. Then the root agent's cost is

$$
C_\emptyset(A, T, T_\emptyset) = (r+k)X_\emptyset + v(A)Y_\emptyset + k\tilde{X}_\emptyset.
$$

The expected cost function is then

$$
\begin{aligned}
\bar{C}_\emptyset(A, T, T_\emptyset) &\equiv E[C_\emptyset(A, T, T_\emptyset)] \\
&= (r+k)[1 - F_S(T_\emptyset)](1-p)e^{-\lambda q h} + (k + (v(A) - k)F_S(T_\emptyset - A))[1 - (1-p)e^{-\lambda q h}].
\end{aligned}
$$

To simplify the notation we introduce the constant $c$ and the function $\ell : \mathcal{A} \to \mathbb{R}^+$.

$$
\begin{aligned}
c &\equiv r + k \\
\ell(A) &\equiv v(A) - k
\end{aligned}
$$

As will become clear in the equilibrium analysis, there are no Nash equilibria for $A < v^{-1}(k)$. Furthermore if $v(A)$ is constant on any interval, all subsequent results hold with slight modification. Thus without loss of generality we assume $\ell(A)$ to be differentiable and strictly monotonically increasing with $\ell(0) = 0$. Again, we will often suppress the dependence on $A$ for the sake of notational simplicity. This gives us

$$\bar{C_\varnothing}(A, T, T_\varnothing) = c[1 - F_S(T_\varnothing)](1 - p)e^{-\lambda qh} + (k + \ell(A)F_S(T_\varnothing - A))[1 - (1 - p)e^{-\lambda qh}]. \quad (7)$$

It is important to notice that the $Y_k, k = 1, 2, ..., N$ still satisfy (1)-(3) while only (1) is valid for the root. Thus from the point of view of an individual agent, if he changes his own threshold, $T_\varnothing \neq T$, it will not change the value of $h$ for the $Y_k,\ k = 1, 2, 3, ..., N$. In particular we have

$$\frac{\partial h}{\partial T_\varnothing} = 0.$$

Thus the best response for the deviant root agent is

$$\sigma_\varnothing(A, T) = \arg\min_{T_\varnothing}\{c[1 - F_S(T_\varnothing)](1 - p)e^{-\lambda qh} + \ell F_S(T_\varnothing - A)[1 - (1 - p)e^{-\lambda qh}]\}.$$

Taking the first derivative we then have

$$\frac{\partial \bar{C}_\varnothing}{\partial T_\varnothing} = -cf_S(T_\varnothing)(1 - p)e^{-\lambda qh} + \ell f_S(T_\varnothing - A)[1 - (1 - p)e^{-\lambda qh}]. \quad (8)$$

Notice that since $f_S(T_\varnothing - A) = 0$ for all $T_\varnothing < A$ we have $\frac{\partial C}{\partial T_\varnothing} < 0$ for all $T_\varnothing \in (0, A)$. Thus any global minima will be in the interval $[A, \infty)$. Using (8) we define two functions, $H(A, T_\varnothing)$ and $V(A, T)$.

$$H(A, T_\varnothing) = \frac{f_S(T_\varnothing - A)}{f_S(T_\varnothing)} \quad (9)$$

$$V(A, T) = \frac{c}{\ell(A)} \frac{(1 - p)e^{-\lambda qh}}{1 - (1 - p)e^{-\lambda qh}} \quad (10)$$

From (8) it is not difficult to see that the single function $V(A, T) - H(A, T_\varnothing)$ has the same sign as $\frac{\partial C_\varnothing}{\partial T_\varnothing}$. As such we have the relations

$$\frac{\partial C}{\partial T_\varnothing} \stackrel{<}{\geqq} 0 \iff H(A, T_\varnothing) \stackrel{<}{\geqq} V(A, T). \quad (11)$$

It is important to note that $H(A, T_\varnothing)$ depends on $A$ and $T_\varnothing$ only, while $V(A, T)$ depends on $A$ and $T$ only. Thus for fixed $A$ and $T$ the functional form of $H(A, T_\varnothing)$ alone will determine the optimal response of a deviant agent. In particular if $H(A, T_\varnothing)$ is non-decreasing then $\bar{C}_\emptyset(A, T, T_\emptyset)$ will be quasi-convex in $T_\emptyset$. However as the network threshold $T$ varies so too will the optimal response of a deviant agent. To better understand how the functions $V(A, T)$ and $H(A, T_\emptyset)$ affect a deviant agent's best response, consider the partial derivative of $V(A, T)$ with respect to $T$.

$$\frac{\partial V}{\partial T} = -\frac{c}{\ell} \frac{\lambda q(1-p)e^{-\lambda q h} \frac{\partial h}{\partial T}}{[1-(1-p)e^{-\lambda q h}]^2} \leq 0.$$

Thus $V(A, T)$ is a non-increasing function of $T$ and strictly monotonically decreasing for $T > A > 0$. We wish to show there is a unique $T^*$ such that $T^* \in \sigma_\emptyset(A, T^*)$, i.e. there exists a unique pure symmetric Nash Equilibrium among agents in the network. It turns out the monotonicity of $H(A, T_\emptyset)$ is sufficient to guarantee this.

**Proposition 4.** *For fixed $A \in \mathcal{A}$ if $H(A, T_\emptyset)$ is non-decreasing then there exists a unique network threshold $T^*$ such that $T^* \in \sigma_\emptyset(A, T^*)$, i.e. there is a unique symmetric, pure NE among agents in the network.*

*Proof.* Fix $A \in \mathcal{A}$. Since $V(A, T)$ is strictly monotonically decreasing for $T > A$ and $H(A, T_\emptyset)$ is non-decreasing, then three possibilities exist: 1) there exists a unique value $\tilde{T} \in [A, \infty)$ such that $H(A, \tilde{T}) = V(A, \tilde{T})$, 2) $H(A, T_\emptyset) < V(A, T)$ for all $T_\emptyset, T \geq A$, and 3) $H(A, T_\emptyset) > V(A, T)$ for all $T_\emptyset, T \geq A$. Suppose the first case is true. Then there exists some values $\epsilon_1, \epsilon_2 \geq 0$ such that $H(A, T_\emptyset) < V(A, \tilde{T})$ for $T_\emptyset < \tilde{T} - \epsilon_1$, $H(A, T_\emptyset) = V(A, \tilde{T})$ for $\tilde{T} - \epsilon_1 \leq T_\emptyset \leq \tilde{T} + \epsilon_2$, and $H(A, T_\emptyset) > V(A, \tilde{T})$ for $T_\emptyset > \tilde{T} + \epsilon_2$. By (11) we have $\sigma_\emptyset(A, \tilde{T}) = [\tilde{T} - \epsilon_1, \tilde{T} + \epsilon_2]$. Clearly $\tilde{T} \in \sigma_\emptyset(A, \tilde{T})$. Furthermore, by the uniqueness of $\tilde{T}$ satisfying $H(A, \tilde{T}) = V(A, \tilde{T})$, it is the only value satisfying $\tilde{T} \in \sigma_\emptyset(A, \tilde{T})$.

Now suppose $H(A, T_\emptyset) < V(A, T)$ for all $T_\emptyset, T > A$. Then 11 implies that $C_\emptyset(A, T, T_\emptyset)$ is monotonically decreasing in $T_\emptyset$ for all $T, T_\emptyset > A$. Thus $\sigma_\emptyset(A, T) = \infty$ for all $A > 0$ and $T > A$. In particular $\sigma_\emptyset(A, \infty) = \infty$ and $T^* = \infty$.

Finally suppose $H(A, T_\emptyset) > V(A, T)$ for all $T_\emptyset, T > A$. Then 11 implies that $C_\emptyset(A, T, T_\emptyset)$ is monotonically increasing in $T_\emptyset$ for all $T, T_\emptyset > A$. Thus $\sigma_\emptyset(A, T) = A$ for all $A > 0$ and $T > A$. In particular $\sigma_\emptyset(A, A) = A$ and $T^* = A$.

□

The following corollary gives us a specific distribution on the noise $S$ which will allow us to find unique, symmetric network equilibria.

**Corollary 2.** *For fixed $A \in \mathcal{A}$ if $S \sim Gamma(\alpha, \beta)$ with $\alpha \geq 1, \beta > 0$, then there exists a unique threshold $T^*$ such that $T^* \in \sigma_\emptyset(A, T^*)$, i.e. there is a unique symmetric, pure NE among agents in the network.*

*Proof.* For fixed $A \in \mathcal{A}$ if $S \sim \text{Gamma}(\alpha, \beta)$ then $f_S(x) \sim x^{\alpha-1}e^{-\beta x}$ for $x \in [0, \infty]$,$\alpha, \beta > 0$.

We then have

$$\begin{aligned} H(A, T_\emptyset) &= \frac{f_S(T_\emptyset - A)}{f_S(T_\emptyset)} \\ &= \frac{(T_\emptyset - A)^{\alpha-1} e^{-\beta(T_\emptyset - A)}}{{T_\emptyset}^{\alpha-1} e^{-\beta T_\emptyset}} \\ &= \left(1 - \frac{A}{T_\emptyset}\right)^{\alpha-1} e^{\beta A}. \end{aligned}$$

Clearly $H(A, T_\emptyset)$ is monotonically increasing in $T_\emptyset$ if $\alpha > 1$ and constant if $\alpha = 1$. The result follows from direct application of Prop. 4. $\square$

We can define a network population best response correspondence $\sigma_p : \mathcal{A} \to \mathcal{T}$ which maps a strategy $A$ to the symmetric, mutual best response among agents in the network for which no individual has an incentive to unilaterally deviate. Prop. 4 and Corr. 2 give sufficient conditions under which $\sigma_p(A)$ is a single valued function.

$$\sigma_p(A) = \begin{cases} A & \text{if } H(A,T) > V(A,T) \text{ for all } T \\ T^* & \text{if } T^* \text{ is the unique solution to } H(A,T^*) = V(A,T^*) \\ +\infty & \text{if } H(A,T) < V(A,T) \text{ for all } T \end{cases}$$

### 3.1.2 Bot Master Utility

The bot master maximizes his utility when he gets the most expected functionality out of the network. His utility will then depend on the fraction of agents that are infected, say $\zeta$, as well as the degree to which he utilizes the bots, which we measure by his aggressiveness, $A$. Since the root agent is chosen uniformly at random from the network it can be thought of as a typical agent. Thus the probability that the root agent is infected is the expected proportion of infected agents. Then in the limit of a large population we have $\zeta = h$. On the infinite Poisson Tree the bot master's expected utility is

$$U(A,T) = g(A)\zeta = g(A)h(A,T). \tag{12}$$

As in the two-person game we assume $g : \mathcal{A} \to \mathbb{R}^+$ satisfies $\frac{dg}{dA} > 0$ and $\frac{d^2g}{dA^2} \leq 0$ with $g(0) = 0$. Thus the set of bet responses, $\sigma_b(T)$, for the botmaster is given by

$$\sigma_b(T) = \arg\max_A \{g(A)h(A,T)\}.$$

Because we are interested in finding pure Nash Equilibria we wish to find under which conditions $U(A,T)$ is strictly quasi-concave and $\sigma_b(T)$ is single valued. Using (4) and (5) a first order optimality condition for a strategy $A^* \in \mathcal{A}$ can be expressed as

$$\frac{g(A^*)}{g'(A^*)} = \frac{F_S(T - A^*)}{f_S(T - A^*)}\theta(A^*, T). \tag{13}$$

Notice we have used the definition for $\theta(A,T)$ from Lemma 1. The following proposition gives a sufficient condition for the strict quasi-concavity of the bot master's expected utility function.

**Proposition 5.** *For $T > 0$, if there exists a unique $A^* \in \mathcal{A}$ satisfying (13), then $U(A,T)$ is strictly quasi-concave with a maximum at $A^*$ and $\sigma_b(T) = A^*$. If $T = 0$ then $U(A,T) = 0$ for all $A \in \mathcal{A}$ and $\sigma_b(0) \equiv \mathcal{A}$.*

*Proof.* Let $T > 0$. Since $g(0) = 0$ we have $U(0,T) = 0$. Furthermore $U(A,T) = 0$ for all $A \geq T$ and $U(A,T) > 0$ for $A \in (0,T)$. Since $h(A,T)$ is a differentiable function in $A$, so too is $U(A,T)$. Thus by Rolle's Theorem there exists at least one $A^*$ in the open interval $(0,T)$ such that $\frac{\partial U}{\partial A}\big|_{A=A^*} = 0$. In addition since $U(0,T) = U(T,T) = 0$ and $U(A,T) > 0$ for $A \in (0,T)$, there must be at least one global maximum in the open interval $(0,T)$ for $T > 0$. Clearly if (13) has a unique solution $A^*$ we must have $U(A^*,T)$ as a global maximum. Consequently $A^*$ is the unique optimal response to the strategy $T$ giving $\sigma_b(T) = A^*$. Now let $T = 0$. Since $h(A,T) = 0$ when $A \geq T$ we get $U(A,T) = 0$ for all $A$. Consequently $A \in \sigma_b(0)$ for all $A \in \mathcal{A}$.

□

We now state a sufficient condition on $F_S(\cdot)$ that guarantee the strict-quasiconcavity of the botmaster's utility function for $T > 0$.

**Proposition 6.** *For fixed $T > 0$, if $\frac{\partial}{\partial A}\left[\frac{F_S(T-A)}{f_S(T-A)}\right] < 1$ for all $A < T$ then (13) has a unique solution $A^* \in (0,T)$.*

*Proof.* We begin by establishing the following facts.

$$\frac{d}{dA}\left[\frac{g(A)}{g'(A)}\right] \geq 1 \tag{14}$$

$$\frac{\partial}{\partial A}\left[\frac{F_S(T-A)}{f_S(T-A)}\theta(A,T)\right] < 1 \tag{15}$$

Differentiating $\frac{g(A)}{g'(A)}$ we obtain

$$\frac{d}{dA}\left[\frac{g(A)}{g'(A)}\right] = 1 - \frac{g(A)g''(A)}{g(A)^2} \geq 1.$$

Differentiating $\frac{F_S(T-A)}{f_S(T-A)}\theta(A,T)$ we obtain

$$\frac{\partial}{\partial A}\left[\frac{F_S(T-A)}{f_S(T-A)}\theta(A,T)\right] = \frac{\partial}{\partial A}\left[\frac{F_S(T-A)}{f_S(T-A)}\right]\theta(A,T) + \left[\frac{1-\lambda q F_S(T-A)}{\theta(A,T)}\right](1-\theta(A,T)).$$

By assumption $\frac{\partial}{\partial A}\left[\frac{F_S(T-A)}{f_S(T-A)}\right] < 1$ and by Lemma 1 we have $0 < \theta(A,T) \le 1$. It remains to be shown $\left[\frac{1-\lambda q F_S(T-A)}{\theta(A,T)}\right] \le 1$. If $F_S(T-A) \ge \frac{1}{\lambda q}$ then this is clearly the case. On the other hand if $F_S(T-A) < \frac{1}{\lambda q}$ then

$$1 - \lambda q F_S(T-A) < 1 - F_S(T-A)(1-p)\lambda q e^{-\lambda q h} = \theta(A,T),$$

giving us $\left[\frac{1-\lambda q F_S(T-A)}{\theta(A,T)}\right] < 1$ from which we obtain $\frac{\partial}{\partial A}\left[\frac{F_S(T-A)}{f_S(T-A)}\theta(A,T)\right] < 1$.

Since $\frac{g(0)}{g'(0)} = 0$ and $\frac{F_S(T)}{f_S(T)}\theta(0,T) > 0$ properties (14) and (15) guarantee that (13) has a unique solution. By Prop. 5 the result follows.

□

The following Corollary gives a particular distribution that guarantees the strict quasi-concavity of $U(A,T)$ in $A$ for $T > 0$.

**Corollary 3.** *If $S \sim$ Gamma$(\alpha,\beta)$ with $\alpha > 0$ and $\beta > 0$ then the botmaster's utility function is strictly quasi-concave in the large population network infection game.*

*Proof.* See Appendix A.2 □

Recall that the gamma distribution contains the exponential distribution as a special case ($\alpha = 1$).

### 3.2 Nash Equilibrium

We are now ready to prove existence and uniqueness of a pure, symmetric Nash equilibria for the large-population botnet detection game with $S \sim \text{Gamma}(\alpha,\beta)$ with $\alpha \ge 1, \beta > 0$ and $\mathcal{A} = \mathcal{T} = \mathbb{R}^+$. As we have seen the function $V(A,T) = \frac{c}{\ell(A)}\frac{(1-p)e^{-\lambda q h(A,T)}}{1-(1-p)e^{-\lambda q h(A,T)}}$ played an important role in determining the response functions of the agents. In the proceeding analysis we will need the limiting values of this function. For ease of exposition we define the functions $V_0 : \mathcal{A} \to \mathbb{R}^+$ and $V_\infty : \mathcal{A} \to \mathbb{R}^+$.

$$V_0(A) \equiv \lim_{T \downarrow A} V(A,T) = \frac{c}{\ell(A)}\frac{1-p}{p}, \tag{16}$$

$$V_\infty(A) \equiv \lim_{T \to \infty} V(A,T) = \frac{c}{\ell(A)}\frac{(1-p)e^{-\lambda q h_\infty}}{1-(1-p)e^{-\lambda q h_\infty}}, \tag{17}$$

where $h_\infty = \lim_{T\to\infty} h(A,T)$ satisfies (6). In what follows we assume the function $\ell(A)$ is unbounded as $A \to \infty$. The main ideas behind the proofs are valid with slight modification for the case of a bounded $\ell(\cdot)$. The monotonicity of $\ell(A)$ gives us the following Lemma.

**Lemma 2.** *Let $V_0(a)$ and $V_\infty(a)$ be defined as in (16) and (17). Then we have the following.*

1. $V_\infty(a) < V_0(a)$ *for all* $a \in (0, \infty)$

2. $\frac{dV_0}{da} = -\frac{\ell'(a)}{\ell(a)} V_0(a) < 0$ *for all* $a \in (0, \infty)$

3. $\frac{dV_\infty}{da} = -\frac{\ell'(a)}{\ell(a)} V_\infty(a) < 0$ *for all* $a \in (0, \infty)$

4. $\lim_{a \downarrow 0} V_0(a) = \lim_{a \downarrow 0} V_0(a) = +\infty$

5. $\lim_{a \to \infty} V_0(a) = \lim_{a \to \infty} V_0(a) = 0$

From the monotonicity of both $V_0(a)$ and $V_\infty(a)$ there must exist unique values $A_0$ and $A_\infty$ satisfying

$$A_0 = \frac{1}{\beta} \log V_0(A_0), \tag{18}$$

$$A_\infty = \frac{1}{\beta} \log V_\infty(A_\infty). \tag{19}$$

To establish our result we will need the following lemmas which give us important properties of $\sigma_b(T)$ and $\sigma_p(A)$.

**Lemma 3.** *Given the expected cost function* $C_\emptyset(A, T)$ *in (7) with* $S \sim gamma(\alpha, \beta)$, $\alpha \geq 1$, *the following properties of* $\sigma_p(A)$ *hold.*

1. *For* $A \geq 0$, $\sigma_p(A) \geq A$.
2. *For* $0 \leq A \leq A_\infty$, $\sigma_p(A) = \infty$.
3. *For* $A > A_\infty$, $\sigma_p(A)$ *is continuously differentiable with* $\lim_{A \downarrow A_\infty} \sigma_p(A) = \infty$.
4. *Let* $A \geq A_0$. *Then for* $\alpha = 1$ *we have* $\sigma_p(A) = A$, *and for* $\alpha > 1$ *we have* $\sigma_p(A) > A$ *with* $\lim_{A \to \infty} \sigma_p(A) - A = 0$.

*Proof.* See Appendix A.3 □

**Lemma 4.** *Given the expected utility function* $U(A, T)$ *in (12) with* $F_S(\cdot)$ *satisfying the properties of Lemma 6, the following properties of* $\sigma_b(T)$ *hold.*

1. *For* $T > 0$, $0 < \sigma_b(T) < T$.
2. *For* $T > 0$, $\sigma_b(T)$ *is continuously differentiable.*
3. $\limsup_{T \to \infty} \sigma_b(T) = \infty$ *with*
   $\limsup_{T \to \infty} (T - \sigma_b(T)) > 0$.

4. *For all $A \in (0, \infty)$ there exists a finite $\tilde{T} > 0$ such that $\sigma_b(\tilde{T}) = A$.*

*Proof.* See Appendix A.4

□

Using the previous technical lemmas we can establish one final lemma which will help us prove the uniqueness of a Nash equilibrium.

**Lemma 5.** *For any $(A, T) \in \mathbb{R}^2$ with $A_\infty < A < T$ we have*

$$\frac{d}{dA}\left[\sigma_p(A) - A\right] < 0 < \frac{d}{dT}\left[T - \sigma_b(T)\right].$$

*Or equivalently*

$$\frac{d\sigma_p}{dA} < 1,$$
$$\frac{d\sigma_b}{dT} < 1.$$

*Proof.* See Appendix A.5

□

We are now ready to state and prove the first main theorem of this paper, namely the existence and uniqueness of a pure, symmetric Nash equilibrium in the infinite-population botnet game on $T(\lambda)$. An analogous result is obtained for a centrally planned network in Appendix B. The results are extended to $G(n, \lambda/n)$ in Appendix C.

**Theorem 1.** *Let $S \sim$ gamma$(\alpha, \beta)$ with $\alpha \geq 1, \beta > 0$ in the infinite-population botnet game on $T(\lambda)$ with homogeneous agents. Then there exists a unique, pure, symmetric Nash equilibrium, i.e. there exists a unique point $(A^*, T^*) \in \mathbb{R}^2$ such that*

$$\sigma_b(T^*) = A^*$$
$$\sigma_p(A^*) = T^*.$$

*Proof.* First consider the special case where $\alpha = 1$ ($S \sim \exp(\beta)$). Suppose $A \leq A_\infty$. It follows from Lemma 3 property 2 that for all $A \leq A_\infty$ we have $\sigma_p(A) = \infty$, hence $U(A, \sigma_p(A)) = \lim_{T\to\infty} Ah(A, T) = Ah_\infty$. But for any finite $M > 0$ we have $U(A + M, T^*) = (A + M)h_\infty > Ah_\infty = U(A, T^*)$ for all such $A$. It follows that $A$ is not a best response and there are no pure Nash equilibrium with $A^* \in [0, A_\infty]$. On the other hand suppose $A \geq A_0$. By Lemma 3 property 4 $\sigma_p(A) = A$. Then we have $U(A, \sigma_p(A)) = U(A, A) = Ah(A, A) = 0$, and for sufficiently small $\epsilon > 0$ we have $U(A - \epsilon, A) = (A - \epsilon)h(A - \epsilon, A) > 0$ and the will benefit from decreasing his aggressiveness . Clearly such an $A$ is not a best response, and any strategy set $(A^*, T^*)$ with $A^* \geq A_0$ is not a Nash equilibrium.

We thus restrict our attention to $A \in (A_\infty, A_0)$. By property 4 of Lemma 4 there exists a finite value $T_\infty$ such that $\sigma_b(T_\infty) = A_\infty$ and a finite value $T_0 > 0$ such that $\sigma_b(T_0) = A_0$. By Lemma 3 we have $\sigma_p(\sigma_b(T_\infty)) = \infty > T_\infty$ and $\sigma_p(\sigma_b(T_0)) = \sigma_b(T_0) < T_0$. In other words when looking along the $T$ axis at $T_\infty$ the function $\sigma_p(\cdot)$ is above the function $\sigma_b(\cdot)$ while at $T_0$ the function $\sigma_p(\cdot)$ is below the function $\sigma_b(\cdot)$. By the continuity of both $\sigma_p(\cdot)$ and $\sigma_b(\cdot)$ the functions must cross at some point $(A^*, T^*)$ giving us $\sigma_b(T^*) = A^*$ and $\sigma_p(A^*) = T^*$.

The proof for $\alpha > 1$ is similar to the above with one exception. In this case we have $\sigma_p(A) > A$ for all $A$ with $\lim_{A\to\infty} \sigma_p(A) = A$. Thus the continuity of $\sigma_p(A)$ and $\sigma_b(T)$ is not enough to guarantee the response functions cross.

Suppose $\sigma_b(\cdot)$ and $\sigma_p(\cdot)$ do not cross. From property 1 of Lemma 4 we have $\lim_{T\to 0} \sigma_b(T) = 0$. Thus there must exist some finite $T_\infty$ such that $0 < \sigma_b(T_\infty) < A_\infty$. Since $\sigma_p(A) = \infty$ for all $A \le A_\infty$ we then have $\sigma_p(\sigma_b(T_\infty)) = \infty$. Thus $\sigma_p(\sigma_b(T_\infty)) > T_\infty$. By our assumption that $\sigma_b(\cdot)$ and $\sigma_p(\cdot)$ do not cross we must have $\sigma_p(\sigma_b(T)) > T$ for all $T > 0$. From Lemma 4 we have $T > \sigma_b(T)$ for all $T > 0$. Together this gives us the following.

$$\sigma_p(\sigma_b(T)) > T > \sigma_b(T) \tag{20}$$

Recall from the above Lemmas that $\limsup_{T\to\infty} \sigma_b(T) = \infty$ and $\lim_{A\to\infty} \sigma_p(A) - A = 0$. It follows that

$$\limsup_{T\to\infty} [\sigma_p(\sigma_b(T)) - \sigma_b(T)] = \lim_{A\to\infty} [\sigma_p(A) - A] = 0.$$

Then by (20) $\limsup_{T\to\infty}[T - \sigma_b(T)] = 0$. But this contradicts $\limsup_{T\to\infty}[T - \sigma_b(T)] > 0$ from Lemma 4. Hence $\sigma_b(\cdot)$ and $\sigma_p(\cdot)$ must cross at least once.

Suppose there is more than one point at which $\sigma_p(A)$ and $\sigma_b(T)$ cross. Let $(A_1^*, T_1^*)$ and $(A_2^*, T_2^*)$ be two such points. We then have for $i = 1, 2$

$$(A_i^*, \sigma_p(A_i^*)) = (\sigma_b(T_i^*), T_i^*). \tag{21}$$

Any points satisfying (21) must also satisfy

$$\sigma_p(A_i^*) - A_i^* = T_i^* - \sigma_b(T_i^*). \tag{22}$$

But by Lemma 5 there is exactly one point satisfying (22). Hence $(A_1^*, T_1^*) = (A_2^*, T_2^*)$ and the equilibrium point is unique.

□

From the proof of the above theorem we can obtain bounds on where the equilibrium will be. We state these bounds as a corollary to the theorem.

**Corollary 4.** *Let $(A^*, T^*)$ be the unique, pure, symmetric Nash equilibrium in Theorem 1. Then we have $T^* \ge \max\{A^*, \frac{\alpha-1}{\beta}\}$ and $A^* \ge \max\{0, A_\infty\}$. In the special case $\alpha = 1$,* *i.e.* $S_i \sim exp(\beta)$, *we have*

$$\max\{0, A_\infty\} \le A^* \le \max\{0, A_0\},$$
$$A^* \le T^* < \inf\left\{T : \sigma_b(T) = \frac{c}{\ell(A_0)}\frac{1-p}{p}\right\}$$

# 4 Discussion and Numerical Examples

Having established the existence and uniqueness of a pure, symmetric Nash equilibrium in both the centralized and decentralized game, we examine some numerical examples to study the efficiency of the equilibria. We observe that the infectivity of the graph, i.e. the parameter $\lambda q$, controls the relative efficiency of the Nash equilibria. Recall that $\lambda$ is the average number of neighbors in the underlying random graph while $q$ is the probability of contagion between neighbors given the presence of the infection. The parameters $\lambda$ and $q$ only appear together as $\lambda q$ in the model, thus we treat them as a single parameter in the numerical approximations.

Figures 5a and 5b show the expected cost and expected utility at equilibrium for varying values of the parameter $\lambda q$. What we find is that for smaller values of $\lambda q$ the centralized planner is worse off than the decentralized agents, while for larger values of $\lambda q$ the centralized planner fares better. More specifically for a given $\lambda q$ let $(A_c^*(\lambda q), T_c^*(\lambda q))$ be the unique pure, symmetric equilibrium of the game between centralized planner and bot master and let $(A_d^*(\lambda q), T_d^*(\lambda q))$ be the unique pure, symmetric equilibrium of the game between decentralized agents and bot master. Define the relative welfare of the network for a given $\lambda q$ as

$$W(\lambda q) \equiv \frac{C(A_c^*(\lambda q), T_c^*(\lambda q))}{C_\emptyset(A_d^*(\lambda q), T_d^*(\lambda q))}.$$

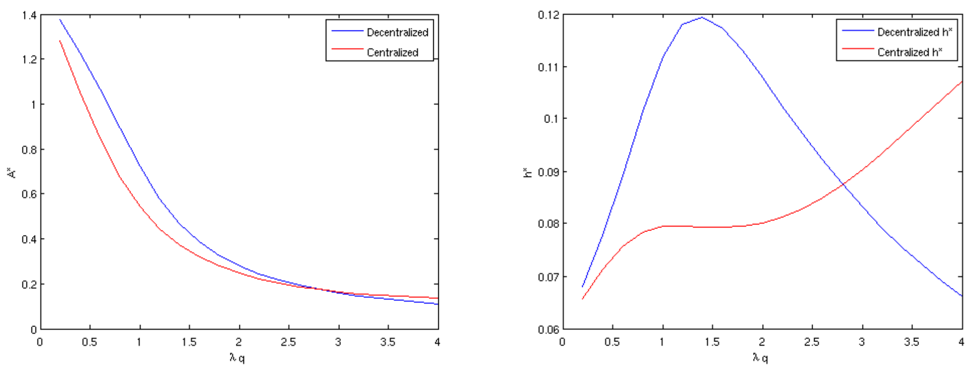
Numerical results suggest the existence of a threshold parameter $\Lambda > 0$ such that

$$W(\lambda q) > 1 \text{ for } \lambda q < \Lambda$$
$$W(\lambda q) = 1 \text{ for } \lambda q = \Lambda$$
$$W(\lambda q) < 1 \text{ for } \lambda q > \Lambda$$

Furthermore a lower cost at equilibrium is associated with a *less* vigilance and *higher* infection rates. For example when the centralized planner has a lower cost than the decentralized agents at equilibrium for a fixed value of $\lambda q$, i.e. for $W(\lambda q) < 1$, the centralized planner is less vigilant, i.e. $T_c^*(\lambda q) > T_d^*(\lambda q)$, and actually admits a higher infection rate than do the decentralized agents, i.e. $h(A_c^*(\lambda q), T_c^*(\lambda q)) > h(A_d^*(\lambda q), T_d^*(\lambda q))$. Conversely, when $W(\lambda q) > 1$ we have $T_c^*(\lambda q) < T_d^*(\lambda q)$ and $h(A_c^*(\lambda q), T_c^*(\lambda q)) < h(A_d^*(\lambda q), T_d^*(\lambda q))$. See Fig. 2 for a plots of $h(A_c^*(\lambda q), T_c^*(\lambda q))$ and $h(A_d^*(\lambda q), T_d^*(\lambda q))$ and Figs. 4a and 4b for plots of the equilibrium strategies $A_c^*, A_d^*, T_c^*, T_d^*$ as functions of the parameter $\lambda q$.

Unless otherwise noted the following values were used for the numerical examples.

$$\begin{aligned} p &= 0.1 \\ \lambda q &\in [0, 4] \\ c &= 1 \\ \ell &= 2 \\ S_i &\sim \exp(1) \end{aligned}$$

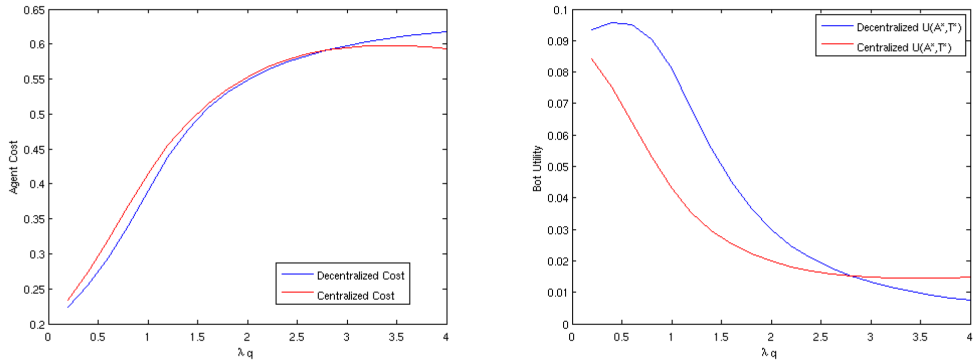



(a) $A^*$ (b) $T^*$

Figure 4: Optimal strategies $A^*$ and $T^*$ at equilibrium as functions of $\lambda q$

# 5 Conclusion

We have considered a novel network security game between a bot master and decentralized agents in a targeted network as well as a game between a bot master and a centralized planner. The game considers the interactions between a bot master's aggressiveness in utilizing his botnet and the degree of vigilance exercised by the agents in a targeted network. To deal with the complexity of the inter-connected nature of botnets we utilized a local mean field model similar to the one developed in [11] . We were able to show the existence and uniqueness of a pure, symmetric Nash equilibrium among agents as well as the existence and uniqueness a pure Nash Equilibrium between the agents and the botmaster. Our analytical results are valid for infinite rooted Poisson Trees and in the limit of a sequence of rooted Erdos-Renyi random graphs. Numerical approximations of expected cost and utility functions suggest some counter intuitive consequences. In particular when contrasting decentralized agents with a centralized planner, better outcomes (lower costs) are associated




(a) $C_\emptyset(A^*, T^*)$ and $C(A^*, T^*)$ (b) $U(A^*, T^*)$

Figure 5: Expected costs $C_\emptyset(A^*, T^*)$, $C(A^*, T^*)$ and expected utility $U(A^*, T^*)$ at equilibrium for varying value of $\lambda q$

with higher infection rates. Furthermore the centralized planner does not always fare better than the decentralized agents. The network infection parameter $\lambda q$ appears to exhibit a threshold between regions where the centralized planner fares better and regions where the decentralized agents do better. In particular we postulate the existence of a threshold parameter $\Lambda$, such that when $\lambda q < \Lambda$ decentralized agents fare better, while $\lambda q > \Lambda$ implies the centralized planner fares better.

Up to this point we have made several simplifying assumptions about the system under consideration in order to obtain a tractable, analytical model. There are several open areas of research which we are currently pursing, which address address some of the shortcomings of our model. One extension under consideration is that of heterogeneous agents. The assumption of heterogeneity is overly restrictive, especially when dealing with social networks and/or internet topology. Examining larger classes of equilibria, in particular mixed equilibria, are also being considered. That a true positive excludes the possibility of contagion is also restrictive and work has been done on weakening this assumption.

# A Appendix: Technical Proofs

## A.1 Proof of Lemma 1

To prove the lemma we fix $T \geq A \geq 0$ and show that for any value of $\lambda q > 0$ the inequality holds. First note that if $T = A$ then $\theta(A, T, \lambda q) = 1$ for all $\lambda q > 0$. Thus we fix $T > A$. We begin by noting that for fixed $A$ and $T$ the Implicit Function Theorem gives us that $h(\lambda q)$ is a differentiable function of $\lambda q$. Furthermore we can show that $h(\lambda q)$ is monotonically increasing in $\lambda q$ as done in [11].From the definition of $h(\lambda q)$ in Prop. 2 it suffices to prove that $\lambda q(F_S(T - A) - h(\lambda q)) < 1$, or equivalently $F_S(T - A) - \frac{1}{\lambda q} < h(\lambda q)$, for all $\lambda q > 0$.

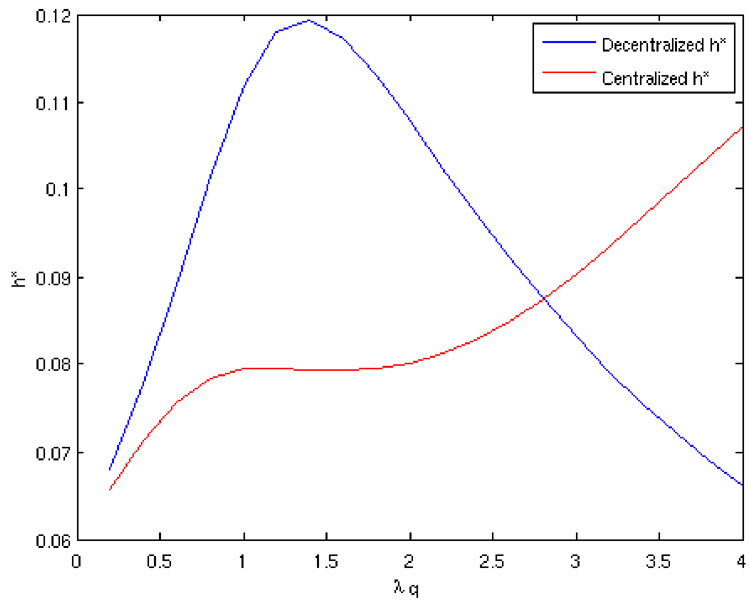



Figure 6: The probability of infection $h(A^*, T^*)$ as a function $\lambda q$. Notice how the centralized planner actually allows the probability of infection to rise as the graph becomes more dense.

The definition in Prop. 2 gives us $F_S(T-A)p \leq h(\lambda q)$. Now, if $0 < \lambda q < \frac{1}{F_S(T-A)(1-p)}$, then

$$\begin{aligned} h(\lambda q) &\geq F_S(T-A)p \\ &= F_S(T-A) - F_S(T-A)(1-p) \\ &> F_S(T-A) - \frac{1}{\lambda q}. \end{aligned}$$

It follows that the claim is true for all $\lambda q \in (0, \frac{1}{F_S(T-A)(1-p)})$. Now suppose there exists a value $y^* \geq \frac{1}{F_S(T-A)(1-p)}$ such that $F_S(T-A) - \frac{1}{y^*} = h(y^*)$. Using the definition of $h$ we have

$$\begin{aligned} F_S(T-A) - \frac{1}{y^*} &= F_S(T-A)[1-(1-p)e^{-y^*(F_S(T-A)-\frac{1}{y^*})}] \\ &= F_S(T-A)[1-(1-p)e^{1-y^*F_S(T-A)}], \end{aligned}$$

which gives

$$1 = y^* F_S(T-A)(1-p)e^{1-y^*F_S(T-A)}.$$

It is straightforward to show that for values $0 < \alpha, \beta < 1$ we must have $\alpha\beta x e^{1-\alpha x} < 1$ for all $x > 0$. But this contradicts our result. Hence no such $y^*$ exists. By the continuity of $h$ in $\lambda q$ there are also no values of $\lambda q$ such that $F_S(T-A) - \frac{1}{\lambda q} > h(\lambda q)$. This establishes our result for fixed $A$ and $T$. Since the choice of $A$ and $T$ was arbitrary this establishes the proposition.

### A.2 Proof of Corollary 3

Note that

$$\frac{\partial}{\partial A}\left[\frac{F_S(T-A)}{f_S(T-A)}\right] = \frac{F_S(T-A)f'_S(T-A)}{[f_S(T-A)]^2} - 1$$

where $f'_S(T-A) = \left.\frac{df_S(x)}{dx}\right|_{x=T-A}$. We will show $\frac{F_S(T-A)f'_S(T-A)}{[f_S(T-A)]^2} < 1$ which implies $\frac{\partial}{\partial A}\left[\frac{F_S(T-A)}{f_S(T-A)}\right] < 0$, giving us the result by Prop. 6. Since the function only depends on $T-A$ we can do a change of variables $x = T-A$. It is thus enough to show that $\frac{F_S(x)f'_S(x)}{[f_S(x)]^2} < 1$ for all $x \in [0, T)$. Notice that this is trivially true whenever $f'_S(x) \le 0$, so we only need to consider the case where $f'_S(x) > 0$. The density for the gamma distribution is $f_S(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$ for $x \in [0, \infty)$ giving $f'_S(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-2}e^{-\beta x}(\alpha - 1 - \beta x)$. Thus $f'_S(x) \le 0$ if and only if $\alpha - 1 \le \beta x$. In particular if $\alpha \le 1$ we are done. We now restrict our analysis to the cases where $\alpha > 1$ and $x < \frac{\alpha-1}{\beta}$. Define $\phi(x)$ and $\xi(x)$ as follows.

$$\begin{aligned}
\phi(x) &= (\alpha - 1 - \beta x)\int_0^x u^{\alpha-1}e^{-\beta u}du \\
\xi(x) &= x^\alpha e^{-\beta x}
\end{aligned}$$

Differentiating with respect to $x$ we obtain,

$$\begin{aligned}
\phi'(x) &= -\beta\int_0^x u^{\alpha-1}e^{-\beta u}du + (\alpha - 1 - \beta x)x^{\alpha-1}e^{-\beta x}, \\
\xi'(x) &= (\alpha - \beta x)x^{\alpha-1}e^{-\beta x}.
\end{aligned}$$

First note that $\phi(0) = \xi(0) = 0$ and $\phi'(0) = \xi'(0) = 0$. Then for $x > 0$ we have $\phi'(x) < (\alpha - 1 - \beta x)x^{\alpha-1}e^{-\beta x} < (\alpha - \beta x)x^{\alpha-1}e^{-\beta x} = \xi'(x)$. It follows that $\phi(x) < \xi(x)$ for all $x > 0$. Thus we have

$$F_S(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^x u^{\alpha-1}e^{-\beta u}du < \frac{\beta^\alpha}{\Gamma(\alpha)}\frac{x^\alpha e^{-\beta x}}{\alpha - 1 - \beta x} = \frac{[f_S(x)]^2}{f'_S(x)}.$$

This establishes our result.

### A.3 Proof of Lemma 3

Property 1) is evident from the proof of Prop. 4. Let $A \leq A_\infty$. Using the definitions of $H(A, T_\emptyset)$ and $V(A,T)$ in (9) and (10) respectively, and the monotonicity of $V_\infty(A)$ in Lemma 2 we have the following for all $T, T_\emptyset \in [A, \infty)$.

$$H(A, T_\emptyset) \leq e^{\beta A} \leq e^{\beta A_\infty} = V_\infty(A_\infty) \leq V_\infty(A) \leq V(A,T)$$

This corresponds to case 2) in the proof of Prop. 4 which implies $\sigma_p(A) = \infty$.

Now fix $A > A_\infty$. That $\sigma_p(A)$ is single-valued is established in Prop. 4 and Corr. 2. For any $(a,t) \in \mathbb{R}^2$ define

$$G(a,t) = H(a,t) - V(a,t).$$

Setting $g(a,t) = 0$ and applying the implicit function theorem gives us the existence of a continuously differentiable function $m(a)$ such that $G(a, m(a)) = 0$ for all $a$ in some open neighborhood of $A$. Since $\sigma_p(A)$ is the unique value satisfying $G(A, \sigma_p(A)) = 0$ for all $A > \frac{1}{\beta}\log V_\infty$, we must have $m(A) = \sigma_p(A)$ for all $A > \frac{1}{\beta}\log V_\infty$. Thus $\sigma_p(A)$ is continuously differentiable for all $A > \frac{1}{\beta}\log V_\infty$.

To show $\lim_{A \downarrow A_\infty} \sigma_p(A) = \infty$ it suffices to show that for any $M > 0$ there exists an $\epsilon > 0$ such that $\sigma_p(A) > M$ whenever $0 < A - A_\infty < \epsilon$. From the definition of $\sigma_p(A)$ and the monotonicity and continuity of $H(A,T)$ and $V(A,T)$ in $T$, it suffices to show that for any $M > 0$ there exists $\tilde{\epsilon} > 0$ such that for any $\epsilon < \tilde{\epsilon}$ there exists $\delta > 0$ such that $H(A_\infty + \epsilon, M) > V(A_\infty + \epsilon, M)$ and $H(A_\infty + \epsilon, M + \delta) < V(A_\infty + \epsilon, M + \delta)$.

Fix $M > A_\infty$. Since $V(A_\infty, T)$ is non-increasing in $T$

$$\frac{V(A_\infty, M)}{V_\infty(A_\infty)} = \frac{V(A_\infty, M)}{\lim_{T\to\infty} V(A_\infty, T)} > 1.$$

By the monotonicity of $\ell(A)$ there exists a $\tilde{\epsilon} > 0$ such that

$$e^{\beta\tilde{\epsilon}} \frac{\ell(A_\infty + \tilde{\epsilon})}{\ell(A_\infty)} = \left(\frac{M}{M - A_\infty}\right)^{\alpha-1} \frac{V(A_\infty, M)}{V_\infty(A_\infty)}$$

Now choose any $\epsilon \in (0, \tilde{\epsilon})$. Notice that since $h(A,T)$ depends on $A$ and $T$ only through the difference $T - A$ we have for any $T > A > 0$ and any constant $k$.

$$\begin{aligned}
\frac{V(A, T-k)}{V_\infty(A)} &= \frac{\frac{c}{\ell(A)} \frac{(1-p)e^{-\lambda q h(A,T-k)}}{1-(1-p)e^{-\lambda q h(A,T-k)}}}{\frac{c}{\ell(A)} \frac{(1-p)e^{-\lambda q h_\infty}}{1-(1-p)e^{-\lambda q h_\infty}}} \\
&= \frac{\frac{c}{\ell(A+k)} \frac{(1-p)e^{-\lambda q h(A+k,T)}}{1-(1-p)e^{-\lambda q h(A+k,T)}}}{\frac{c}{\ell(A+k)} \frac{(1-p)e^{-\lambda q h_\infty}}{1-(1-p)e^{-\lambda q h_\infty}}} \\
&= \frac{V(A+k, T)}{V_\infty(A+k)}
\end{aligned}$$

Furthermore $V_\infty(A+k) = \frac{\ell(A_\infty)}{\ell(A_\infty+k)} V_\infty(A_\infty)$ for any constant $k$. We thus have the following.

$$\begin{aligned}
1 &< e^{-\beta\epsilon}\frac{\ell(A_\infty)}{\ell(A_\infty+\epsilon)}\left(\frac{M}{M-A_\infty}\right)^{\alpha-1}\frac{V(A_\infty, M)}{V_\infty(A_\infty)} \\
&< e^{-\beta\epsilon}\frac{\ell(A_\infty)}{\ell(A_\infty+\epsilon)}\left(\frac{M}{M-A_\infty}\right)^{\alpha-1}\frac{V(A_\infty, M-\epsilon)}{V_\infty(A_\infty)} \\
&= e^{-\beta\epsilon}\frac{\ell(A_\infty)}{\ell(A_\infty+\epsilon)}\left(\frac{M}{M-A_\infty}\right)^{\alpha-1}\frac{V(A_\infty+\epsilon, M)}{\frac{\ell(A_\infty)}{\ell(A_\infty+\epsilon)}V_\infty(A_\infty)} \\
&= e^{-\beta(A_\infty+\epsilon)}\left(\frac{M}{M-A_\infty}\right)^{\alpha-1} V(A_\infty+\epsilon, M) \\
&< e^{-\beta(A_\infty+\epsilon)}\left(\frac{M}{M-\epsilon-A_\infty}\right)^{\alpha-1} V(A_\infty+\epsilon, M) \\
&= e^{-\beta(A_\infty+\epsilon)}\left(\frac{M}{M-(A_\infty+\epsilon)}\right)^{\alpha-1} V(A_\infty+\epsilon, M) \\
&= \frac{V(A_\infty+\epsilon, M)}{H(A_\infty+\epsilon, M)}.
\end{aligned}$$

Now choose $\delta > 0$ such that

$$\left(\frac{M+\delta}{M+\delta-\epsilon-A_\infty}\right)^{\alpha-1} V(A_\infty, M+\delta-\epsilon) < e^{\beta\epsilon} V_\infty$$

Note that $\delta$ is guaranteed to exist by the continuity and monotonicity of $V(A,T)$ and $\left(\frac{T}{T-A}\right)^{\alpha-1}$ in $T$ for $T > A$. We now have

$$e^{-\beta\epsilon}\left(\frac{M+\delta}{M+\delta-\epsilon-A_\infty}\right)^{\alpha-1}\frac{V(A_\infty, M+\delta-\epsilon)}{V_\infty} < 1,$$

which is equivalent to $V(A_\infty+\epsilon, M+\delta) < H(A_\infty+\epsilon, M+\delta)$.

Now fix $A \geq A_0$. For $\alpha = 1$ we have for all $T, T_\varnothing > A$

$$H(A, T_\varnothing) = e^{\beta A} \geq e^{\beta A_0} =\geq V_0(A_0) \geq V_0(A) \geq V(A,T).$$

This corresponds to case 1) with $\tilde{T} = A$ or case 3) in the proof of Prop. 4, which implies $\sigma_p(A) = A$ in either case.

For $\alpha > 1$ we have $H(A, T_\varnothing) = \left(1 - \frac{A}{T_\varnothing}\right)^{\alpha-1} e^{\beta A}$. For all $T \geq A$ we have the following.

$$\lim_{T_\varnothing \downarrow A} H(A, T_\varnothing) = 0 \leq V_\infty(A) < V(A,T)$$
$$\lim_{T_\varnothing \to \infty} H(A, T_\varnothing) = e^{\beta A} \geq V_0(A) \geq V(A,T)$$

Since $H(A, T_\emptyset)$ is monotonically increasing and $V(A, T)$ is monotonically decreasing there must exist a $\tilde{T}(A)$ such that $H(A, \tilde{T}) = V(A, \tilde{T})$ and by the proof of Prop. 4 we must have $\sigma_p(A) = \tilde{T}(A)$. Similarly there must exist a $T_0(A)$ such that $H(A, T_0(A)) = V_0(A)$. By the monotonicity of $H(A, T_\emptyset)$ and $V(A, T)$ we must have $A \leq \tilde{T}(A) = \sigma_p(A) \leq T_0(A)$ for all $A \geq A_0$. Since $H(A, T_0(A)) = V_0(A)$ we have

$$\left(1 - \frac{A}{T_0(A)}\right)^{\alpha-1} e^{\beta A} = V_0(A).$$

It follows that

$$T_0(A) - A = \frac{A\left[V_0(A)e^{-\beta A}\right]^{\frac{1}{\alpha-1}}}{1 - \left[V_0(A)e^{-\beta A}\right]^{\frac{1}{\alpha-1}}}.$$

Taking the limit $A \to \infty$ on both sides gives us $\lim_{A\to\infty} T_0(A) - A = 0$. Since $A \leq \sigma_p(A) \leq T_0(A)$ it follows that $\lim_{A\to\infty} \sigma_p(A) - A = 0$.

## A.4 Proof of Lemma 4

That $\sigma_b(T) < T$ for $T > 0$ is apparent in the proof of Prop. 5. For property 2) choose $T > 0$ and for any $(a, t) \in \mathbb{R}^2$ define

$$y(a, t) = G(a, t) - a.$$

Setting $y(a, t) = 0$ and applying the implicit function theorem gives us the existence of a continuously differentiable function $k(t)$ such that $y(k(t), t) = 0$ for all $t$ in some open neighborhood of $T$. By the strict quasi-concavity of $U(A, T)$ established in Prop. 5 we have $\sigma_b(T)$ as the unique value satisfying $y(\sigma_b(T), T) = 0$ for any $T > 0$. Therefore $k(T) = \sigma_b(T)$ for all $T > 0$ and $\sigma_b(T)$ is continuously differentiable for all $T > 0$.

Suppose 3) is false. In particular assume $\limsup_{T\to\infty} \sigma_b(T) < \infty$. Then there exists some value $L > 0$ such that $\sigma_b(T) < L$ for all $T$. By the optimality of $\sigma_b(T)$ we should have $U(\sigma_b(T), T) \geq U(A, T)$ for all $A, T$. However, notice that

$$\limsup_{T\to\infty} U(\sigma_b(T), T) = \limsup_{T\to\infty} \sigma_b(T) h(\sigma_b(T), T) < L h_\infty,$$

but for any $\epsilon > 0$ we have

$$\begin{aligned}\lim_{T\to\infty} U(L + \epsilon, T) &= (L + \epsilon) \lim_{T\to\infty} h(L + \epsilon, T) \\ &= (L + \epsilon) h_\infty > L h_\infty.\end{aligned}$$

It follows that there exists some $T_0$ such that $U(\sigma_b(T_0), T_0) < U(L + \epsilon, T_0)$. This violates the optimality of $\sigma_b(T)$, hence $\limsup_{T\to\infty} \sigma_b(T) = \infty$. Now suppose

$$\limsup_{T\to\infty} (T - \sigma_b(T)) = 0.$$

Then $\limsup_{T\to\infty} \frac{T}{\sigma_b(T)} = 1$ which implies $\limsup_{T\to\infty} \frac{T-1}{\sigma_b(T)} = 1$. Again by the optimality of $\sigma_b(T)$ we should have $\frac{U(\sigma_b(T),T)}{U(A,T)} \geq 1$ for all $A < T$. But

$$\begin{aligned}
\limsup_{T\to\infty} \frac{U(\sigma_b(T),T)}{U(T-1,T)} &= \limsup_{T\to\infty} \frac{U(\sigma_b(T),T)}{U(T-1,T)} \frac{T-1}{\sigma_b(T)} \\
&= \limsup_{T\to\infty} \frac{\frac{U(\sigma_b(T),T)}{\sigma_b(T)}}{\frac{U(T-1,T)}{T-1}} \\
&= \limsup_{T\to\infty} \frac{h(\sigma_b(T),T)}{h(T-1,T)} = 0,
\end{aligned}$$

where the last equality holds since $\limsup_{T\to\infty} h(\sigma_b(T),T) = 0$ by our assumption that $\limsup_{T\to\infty} T - \sigma_b(T) = 0$ while $\limsup_{T\to\infty} h(T-1,T) > 0$. It follows that there exists a $T_0$ such that $\frac{U(\sigma_b(T_0),T_0)}{U(T_0-1,T_0)} < 1$ which violates the optimality of $\sigma_b(T)$.

Property 4) follows from Properties 1)-3).

### A.5 Proof of Lemma 5

For each $T > 0$ the function $\sigma_b(T)$ is the unique solution in $A$ to the equation

$$\frac{g(A)}{g'(A)} = \frac{F_S(T-A)}{f_S(T-A)}\theta(A,T).$$

Define $\tilde{h}, \tilde{\theta}, G : \mathbb{R}^+ \to \mathbb{R}^+$ for $x \in \mathbb{R}^+$ as follows.

$$\begin{aligned}
\tilde{h}(x) &= F_S(x)[1-(1-p)e^{-\lambda q \tilde{h}(x)}] \\
\tilde{\theta}(x) &= 1 - F_S(x)\lambda q(1-p)e^{-\lambda q \tilde{h}(x)} \\
G(x) &= \frac{F_S(x)}{f_S(x)}\tilde{\theta}(x)
\end{aligned}$$

Then for any $T > 0$ the function $\sigma_b(T)$ satisfies the relation

$$\frac{g(\sigma_b(T))}{g'(\sigma_b(T))} = G(T - \sigma_b(T)).$$

Setting $u(T) = T - \sigma_b(T)$ and $y(T) = \sigma_b(T)$ the chain rule gives us the following.

$$\frac{d\sigma_b}{dT} = \frac{\frac{dG}{du}}{\frac{d}{dy}\left[\frac{g(y)}{g'(y)}\right] + \frac{dG}{du}}$$

From Prop. 6 we get $\frac{dG}{dx} > -1$ and $\frac{d}{dy}[\frac{g(y)}{g'(y)}] \geq 1$ . It follows that $\frac{d\sigma_b}{dT} < 1$.

For each $A \geq 0$ the function $\sigma_p(A)$ is defined as the unique solution $T > A$ to the equation

$$\frac{f_S(T-A)}{f_S(T)} = \frac{c}{\ell(A)} \frac{(1-p)e^{-\lambda q h(A,T)}}{1-(1-p)e^{-\lambda q h(A,T)}}.$$

Thus we have for each $A \geq 0$ the relation

$$\frac{f_S(\sigma_p(A)-A)}{f_S(\sigma_p(A))} = \frac{c}{\ell(A)} \frac{(1-p)e^{-\lambda q h(A,\sigma_p(A))}}{1-(1-p)e^{-\lambda q h(A,\sigma_p(A))}}.$$

For notational convenience define the following function.

$$\begin{aligned} g(u) &= \frac{(1-p)e^{-\lambda q h(A,u)}}{1-(1-p)e^{-\lambda q h(A,u)}} \left[ \frac{f_S'(u)}{f-S(u-A)} - \frac{f_S(u)f_S'(u-A)}{f_S(u-A)^2} - f_S(u) \right] \\ &= \frac{d}{dx} \left[ \frac{f_S(x)}{f_S(x-A)} \frac{(1-p)e^{-\lambda q h(A,x)}}{1-(1-p)e^{-\lambda q h(A,x)}} \right]_{x=u} \end{aligned}$$

From Lemma 10 (see Centralized Planner) $g(u) < 0$ for any $u > A$. In particular $g(\sigma_p(A)) < 0$. By the implicit function theorem and chain rule we can write

$$\frac{d\sigma_p}{dA} = 1 + \frac{d\ell}{dA} \frac{1}{cg(\sigma_p(A))} - \frac{f_S'(\sigma_p(A))}{g(\sigma_p(A))f_S(\sigma_p(A)-A)} \frac{(1-p)e^{-\lambda q h(A,\sigma_p(A))}}{1-(1-p)e^{-\lambda q h(A,\sigma_p(A))}} \tag{23}$$

By the properties of $\sigma_p(A)$ from Lemma 4 there must exist a global minimum at some point $A_0$ such that $\left.\frac{d\sigma_p}{dA}\right|_{A=A_0} = 0$ and $\sigma_p(A_0) \leq \sigma_p(A)$ for all $A > 0$. Whenever $\frac{d\sigma_p}{dA} = 0$ we have

$$\frac{f_S'(\sigma_p(A))}{f_S(\sigma_p(A)-A)} \frac{(1-p)e^{-\lambda q h(A,\sigma_p(A))}}{1-(1-p)e^{-\lambda q h(A,\sigma_p(A))}} = g(\sigma_p(A)) + \frac{d\ell}{dA}\frac{1}{c}.$$

Using the definition for $g(\sigma_p(A))$ this gives us the following.

$$\frac{d\ell}{dA}\frac{1}{c} = \frac{(1-p)e^{-\lambda q h(A,u)}}{1-(1-p)e^{-\lambda q h(A,u)}} \left[ -\frac{f_S(\sigma_p(A))f_S'(\sigma_p(A)-A)}{f_S(\sigma_p(A)-A)^2} - f_S(\sigma_p(A)) \right]$$

By assumption $\frac{d\ell}{dA} \geq 0$, thus we must have $f_S'(\sigma_p(A_0) - A_0) < 0$. If $S \sim \text{gamma}(\alpha, \beta)$ then this implies $f_S'(\sigma_p(A_0)) < 0$. Since $f_S(x)$ is either unimodal or monotonically decreasing and $\sigma_p(A_0) \leq \sigma_p(A)$ for all $A > 0$, then we must have $f_S'(\sigma_p(A)) < 0$ for all $A > 0$. It then follows from (23) that $\frac{d\sigma_p}{dA} < 1$. We immediately obtain the desired result, namely

$$\frac{d}{dA}\left[\sigma_p(A) - A\right] < 0 < \frac{d}{dT}\left[T - \sigma_b(T)\right].$$

# B A Centralized, Large-Population Botnet Game

## B.1 Centralized Expected Cost and Best Response

Another approach to the large-population botnet game is to consider the effects of a centralized planner on the game. Suppose there is a single player $P_0$ whose strategy space is $\mathcal{T}$ and chooses a threshold $T \in \mathcal{T}$ for *all* agents to follow. For tractability we set $k \equiv 0$ and assume $\ell(A) \equiv \ell$ is constant. The results hold for non-constant $\ell$ while the case $k > 0$ is an open problem. Otherwise we keep the same expected cost function and best response correspondence as for an individual agent.

$$\begin{aligned} C(A,T) &= c[1 - F_S(T)](1-p)e^{-\lambda q h(A,T)} + \ell F_S(T-A)[1-(1-p)e^{-\lambda q h(A,T)}] \\ \sigma_c(A) &= \arg\min_T \{C(A,T)\} \end{aligned}$$

The difference is that now $\frac{\partial h}{\partial T} > 0$. For ease of exposition we introduce the function $\rho(A,T)$.

$$\rho(A,T) = (1-p)e^{-\lambda q h(A,T)}$$

With this new notation we can rewrite $h(A,T)$ and $\theta(A,T)$.

$$\begin{aligned} h(A,T) &= 1 - F_S(T-A)\rho(A,T) \\ \theta(A,T) &= 1 - F_S(T-A)\lambda q \rho(A,T) \end{aligned}$$

In what follows we assume $A$ is fixed and suppress the dependence on $A$ and $T$ in the notation.

$$\begin{aligned} \rho &= (1-p)e^{-\lambda q h} \\ h &= 1 - F_S(T-A)\rho \\ \theta &= 1 - F_S(T-A)\lambda q \rho \end{aligned}$$

Taking derivatives we obtain

$$\begin{aligned} \frac{\partial \rho}{\partial T} &= -\frac{\partial h}{\partial T}\lambda q \rho = -\frac{f_S(T-A)\lambda q \rho(1-\rho)}{\theta} \\ \frac{\partial h}{\partial T} &= \frac{f_S(T-A)(1-\rho)}{\theta}, \\ \frac{\partial \theta}{\partial T} &= -\frac{f_S(T-A)\lambda q \rho}{\theta}\left[1 - F_S(T-A)\lambda q\right]. \end{aligned}$$

Taking first derivatives of $C(A,T)$ we obtain

$$\begin{aligned} \frac{\partial C}{\partial T} = -c f_S(T)\rho - c[1-F_S(T)]\lambda q \rho \frac{f_S(T-A)(1-\rho)}{\theta} + \\ \ell f_S(T-A)(1-\rho) + \ell F_S(T-A)\lambda q \rho \frac{f_S(T-A)(1-\rho)}{\theta}. \end{aligned}$$

Provided that $f_S(T-A)(1-\rho) > 0$ it follows that

$$\frac{\partial C}{\partial T}\frac{\theta}{\ell f_S(T-A)(1-\rho)} = -\frac{c}{\ell}\frac{f_S(T)}{f_S(T-A)}\frac{\rho}{1-\rho}\theta - \frac{c}{\ell}[1-F_S(T)]\lambda q\rho + \theta + F_S(T-A)\lambda q\rho.$$

But

$$\theta + F_S(T-A)\lambda q\rho = 1.$$

Thus

$$\frac{\partial C}{\partial T}\frac{\theta}{\ell f_S(T-A)(1-\rho)} = -\frac{c}{\ell}\frac{f_S(T)}{f_S(T-A)}\frac{\rho}{1-\rho}\theta - \frac{c}{\ell}[1-F_S(T)]\lambda q\rho + 1.$$

Define the function

$$M(A,T) = -\frac{c}{\ell}\frac{f_S(T)}{f_S(T-A)}\frac{\rho}{1-\rho}\theta - \frac{c}{\ell}[1-F_S(T)]\lambda q\rho + 1.$$

Since $\frac{\theta}{\ell f_S(T-A)(1-\rho)} > 0$ it is clear that the sign of $M(A,T)$ is the same as the sign of $\frac{\partial C}{\partial T}$.
Thus

$$\frac{\partial C}{\partial T} \underset{=}{\overset{<}{>}} 0 \iff M(A,T) \underset{=}{\overset{<}{>}} 0.$$

Alternatively we can define the functions

$$R(A,T) = \frac{c}{\ell}\frac{f_S(T)}{f_S(T-A)}\frac{\rho}{1-\rho}\theta,$$
$$L(A,T) = 1 - \frac{c}{\ell}[1-F_S(T)]\lambda q\rho,$$

so that $M(A,T) = L(A,T) - R(A,T)$ and

$$\frac{\partial C}{\partial T} \underset{=}{\overset{<}{>}} 0 \iff L(A,T) \underset{=}{\overset{<}{>}} R(A,T).$$

It is straight forward to show that $L(A,T)$ is increasing in $T$. It remains to be shown that $R(A,T)$ is decreasing in $T$. To do so we will need the function

$$h_\infty(\phi) = 1 - (1-p)e^{-\phi h_\infty(\phi)}$$

as well as the following technical lemmas.

**Lemma 6.** *Let $h_\infty(\phi)$ be defined as above. Then*

1. $\lim_{\phi\to 0} h_\infty(\phi) = p$
2. $\lim_{\phi\to\infty} h_\infty(\phi) = 1$
3. $\lim_{\phi\to 0} \phi e^{-\phi h_\infty(\phi)} = 0$
4. $\lim_{\phi\to\infty} \phi e^{-\phi h_\infty(\phi)} = 0$

*Proof.* The proofs follows directly from the definition of $h_\infty(\phi)$. $\square$

**Lemma 7.** *For any $\phi \geq 0$, and $p \in [0,1]$ we have*

$$\phi e^{-\phi h_\infty(\phi)} \overset{<}{\geqq} \frac{1}{2(1-p)} \iff \phi e^{-\phi} \overset{<}{\geqq} \frac{e^{-\frac{1}{2}}}{2(1-p)}$$

*Proof.* Define the functions

$$f_\infty(\phi) = 2(1-p)\phi e^{-\phi h_\infty(\phi)},$$
$$y(\phi) = 2(1-p)\phi e^{-\phi+\frac{1}{2}}.$$

To establish our result it suffices to prove that

$$f_\infty(\phi) \overset{<}{\geqq} 1 \iff y(\phi) \overset{<}{\geqq} 1. \tag{24}$$

By the implicit function theorem both $f_\infty(\phi)$ and $y(\phi)$ are differentiable in $\phi$. It is then straight forward to show that both $f_\infty(\phi)$ and $y(\phi)$ are strictly quasi-concave with unique global maxima at $\phi = 1$. Furthermore

$$f_\infty(\phi) < y(\phi) \iff 2(1-p)\phi e^{-\phi h_\infty(\phi)} < 2(1-p)\phi e^{-\phi+\frac{1}{2}} \iff$$
$$-\phi h_\infty(\phi) < -\phi + \frac{1}{2} \iff h_\infty(\phi) > 1 - \frac{1}{2\phi} \iff f_\infty(\phi) < 1.$$

Similarly

$$f_\infty(\phi) \overset{<}{\geqq} y(\phi) \iff f_\infty(\phi) \overset{<}{\geqq} 1. \tag{25}$$

First consider the case $p \geq 1 - \frac{1}{2}e^{\frac{1}{2}}$. It follows that for all $\phi \geq 0$

$$y(\phi) \leq \max_{\phi\geq 0} y(\phi) = y(1) = 2(1-p)e^{-\frac{1}{2}} \leq 1,$$

with $y(\phi) = 1$ if and only if $p = 1 - \frac{1}{2}e^{\frac{1}{2}}$ and $\phi = 1$. Suppose there exists a $\phi_0 > 0$ such that $f_\infty(\phi_0) > 1$. By Lemma 6

$$\lim_{\phi\to 0} f_\infty(\phi) = \lim_{\phi\to 0} 2(1-p)\phi e^{-\phi h_\infty(\phi)} = 0.$$

By the continuity of $f_\infty(\phi)$ there must exist a value $\phi_1 \in (0, \phi_0)$ such that $f_\infty(\phi_1) = 1$. By Eq. 25 this implies $y(\phi_1) = 1$. On the other hand, by Lemma 6

$$\lim_{\phi\to\infty} f_\infty(\phi) = \lim_{\phi\to\infty} 2(1-p)\phi e^{-\phi h_\infty(\phi)} = 0.$$

Again by the continuity of $f_\infty(\phi)$ there must exist a value $\phi_2 > \phi_0 > \phi_1$ such that $f_\infty(\phi_2) = 1$. By Eq. 25 this implies $y(\phi_2) = 1$. But by our assumption $y(\phi) = 1$ if and only if $p = 1 - \frac{1}{2}e^{\frac{1}{2}}$ and $\phi = 1$. This contradicts the result that $y(\phi_1) = y(\phi_2) = 1$ and $\phi_2 > \phi_1$. Thus no such $\phi_0$ exists and we must have $f_\infty(\phi_0) \leq 1$. It is no hard to see that if $p > 1 - \frac{1}{2}e^{\frac{1}{2}}$ then both $y(\phi)$ and $f_\infty(\phi)$ are strictly less than one and Eq. 24 holds. SImilarly if $p = 1 - \frac{1}{2}e^{\frac{1}{2}}$ then $y(\phi)$ and $f_\infty(\phi)$ are strictly less than one if and only if $\phi \neq 1$ and $y(1) = f_\infty(1) = 1$. Again Eq. 24 holds.

Now consider the case $p < 1 - \frac{1}{2}e^{\frac{1}{2}}$. In this case there exist two values $\phi_1$ and $\phi_2$ with $0 < \phi_1 < 1 < \phi_2$ that are solutions to $y(\phi) = 1$. Furthermore by the continuity and strict quasi-concavity of $y(\phi)$ we must have

$$\begin{aligned} y(\phi) > 1 &\iff \phi \in (\phi_1, \phi_2), \\ y(\phi) < 1 &\iff \phi \notin [\phi_1, \phi_2], \\ y(\phi) = 1 &\iff \phi \in \{\phi_1, \phi_2\}. \end{aligned}$$

For $\phi \notin (\phi_1, \phi_2)$ the same contradiction arguments used in the case $p \leq 1 - \frac{1}{2}e^{\frac{1}{2}}$ can be used to establish condition (24). Thus we need only establish the result for $\phi \in (\phi_1, \phi_2)$.

Let $\phi \in (\phi_1, \phi_2)$. Then it must be that $y(\phi) > 1$. Recall that both $y(\phi)$ and $f_\infty(\phi)$ take their maximum values at $\phi = 1$. We claim that for $p < 1 - \frac{1}{2}e^{\frac{1}{2}}$ we have $f_\infty(1) > y(1)$. Let $\tilde{p} = 1 - \frac{1}{2}e^{\frac{1}{2}}$. By the implicit function theorem $h_\infty(1)$ and $y(1)$ are differentiable functions in $p$. It is not hard to show that $h_\infty(1)$ is monotonically increasing in $p$ and $\lim_{p\to 0} h_\infty(1) = 0$. By the continuity and monotonicity of $h_\infty(1)$ in $p$ there exists some value $p_0 \in [0, 1]$ such that $h_\infty(1) = \frac{1}{2}$ when $p = p_0$, $h_\infty(1) < \frac{1}{2}$ when $p < p_0$ and $h_\infty(1) > \frac{1}{2}$ when $p > p_0$. It is straight forward to also show that

$$f_\infty(1) \overset{>}{\underset{=}{\leq}} y(1) \iff h_\infty \overset{<}{\underset{=}{\geq}} \frac{1}{2}.$$

Suppose $p_0 > \tilde{p}$. Then for $p = p_0$ we would have $f_\infty(1) = y(1)$. By (25) this implies $f_\infty(1) = y(1) = 1$, but this contradicts the fact that $y(\phi) < 1$ for all $p > \tilde{p}$. Suppose on the other hand that $p_0 < \tilde{p}$. Then at $p = p_0$ we would have $f_\infty(1) = y(1)$. By (25) this implies $f_\infty(1) = y(1) = 1$, but this contradicts the fact that $y(\phi) > 1$ for all $\phi \in (\phi_1, \phi_2)$ when $p < \tilde{p}$. It follows that $p_0 = \tilde{p}$ and $f_\infty(1) > y(1)$ for all $p < \tilde{p}$.

Now suppose there exists a value $\phi_0 \in (\phi_1, 1)$ such that $f_\infty(\phi_0) < y(\phi_0)$. Since $f_\infty(\phi) < y(\phi)$ for $\phi < \phi_1$ the continuity of $f_\infty(\phi)$ and $y(\phi)$ imply the existence of a point $\phi_3 > \phi_0$ such that $f_\infty(\phi_3) = y(\phi_3)$. Again by (25) this implies $f_\infty(\phi_3) = y(\phi_3) = 1$. This contradicts the fact that $y(\phi) > 1$ for all $\phi \in (\phi_1, \phi_2)$. A similar arguments shows there is no such point in $(1, \phi_2)$. It follows that $f_\infty(\phi) > y(\phi)$ for all $\phi \in (\phi_1, \phi_2)$. Since $y(\phi) > 1$ for all $\phi \in (\phi_1, \phi_2)$ we have $f_\infty(\phi) > 1$. Thus $y(\phi) > 1 \implies f_\infty(\phi) > 1$. The other direction is trivial since $y(\phi) > 1$ for all $\phi \in (\phi_1, \phi_2)$.

$\square$

**Lemma 8.** *For any $\phi \geq 0$, and $p \in [0, 1]$ we have*

$$1 - 2(1-p)\phi e^{-\phi h_\infty(\phi)} + (1-p)e^{-\phi h_\infty(\phi)} \geq 0.$$

*Proof.* It follows from Lemma 7 that

$$1 - 2(1-p)\phi e^{-\phi h_\infty(\phi)} > 0 \iff 1 - 2\phi e^{-\phi + \frac{1}{2}} > 0.$$

Thus if $p \geq 1 - \frac{1}{2}e^{\frac{1}{2}}$ then $1 - 2(1-p)\phi e^{-\phi h_\infty(\phi)} + (1-p)e^{-\phi h_\infty(\phi)} > 0$ and we are done. Now let $p < 1 - \frac{1}{2}e^{\frac{1}{2}}$. Again let $\phi_1$ and $\phi_2$ be solutions to $2\phi e^{-\phi+\frac{1}{2}} = 1$ with $0 < \phi_1 < 1 < \phi_2$. By Lemma 7 if $\phi \notin (\phi_1, \phi_2)$ then $1 - 2(1-p)\phi e^{-\phi h_\infty(\phi)} > 0$ again giving us the result. Now let $\phi \in (\phi_1, \phi_2)$. Then $1 - 2(1-p)\phi e^{-\phi h_\infty(\phi)} < 0$.

Suppose $\phi \in (\phi_1, 1]$. It follows that $(1-p)e^{-\phi h_\infty(\phi)} - (1-p)\phi e^{-\phi h_\infty(\phi)} \geq 0$. At the same time $1 - (1-p)\phi e^{-\phi h_\infty(\phi)} > 0$. Combining these inequalities we arrive at $1 - 2(1-p)\phi e^{-\phi h_\infty(\phi)} + (1-p)e^{-\phi h_\infty(\phi)} > 0$. Now suppose $\phi \in (1, \phi_2)$. Define the function

$$u(\phi) = (2\phi - 1)(1-p)e^{-\phi h_\infty(\phi)},$$

which is clearly differentiable in $\phi$. It suffices to show that $u(\phi) \leq 1$ for $\phi \in (1, \phi_2)$. Notice that $u(1) = (1-p)e^{-h_\infty(1)} < 1$. Suppose there exists a value $\phi_0$ such that $u(\phi_0) > 1$. By continuity there must exist a value $\phi_c \in (1, \phi_0)$ such that $u(\phi_c) = 1$. We then have

$$(2\phi_c - 1)(1-p)e^{-\phi_c h_\infty(\phi_c)} = 1,$$

from which it follows that

$$h_\infty(\phi_c) = 1 - \frac{1}{2\phi_c - 1}.$$

Plugging this into the definition of $h_\infty(\phi)$ gives

$$1 = (2\phi_c - 1)(1-p)e^{-\phi_c + \frac{\phi_c}{2\phi_c - 1}}.$$

Define the function

$$k(\phi) = (2\phi - 1)(1-p)e^{-\phi + \frac{\phi}{2\phi - 1}}.$$

Differentiating we obtain

$$\frac{\partial k}{\partial \phi} = -4\frac{(1-p)e^{-\phi+\frac{\phi}{2\phi-1}}}{2\phi-1}(\phi-1)^2.$$

Notice that for $\phi > 1$ we have $\frac{\partial k}{\partial \phi} < 0$. Thus $\max_{\phi\in(1,\phi_2)} k(\phi) = k(1) = (1-p)e^{-1+\frac{1}{2-1}} = 1-p < 1$. But this contradicts our assumption that $k(\phi_c) = 1$. Thus there is no such $\phi_0$ and we must have $u(\phi) \leq 1$. This establishes our result.

$\square$

**Lemma 9.** *For any $T \geq A$, $\lambda q > 0$ and $p \in [0,1]$, if $\frac{f_S(T-A)}{f_S(T)}$ is non-decreasing in $T$ then*

$$1 - 2F_S(T-A)\lambda q\rho(A,T) + \rho(A,T) \geq 0.$$

*Proof.* Define the function

$$g(A,T) = 1 - 2F_S(T-A)\lambda q\rho(A,T) + \rho(A,T).$$

First note that $g(A,A) = 1 + \rho(A,A) = p > 0$. Furthermore

$$\frac{\partial g}{\partial T} = -\lambda q f_S(T-A)\frac{\rho}{\theta}\left(2(1-\lambda q F_S(T-A)) + (1-\rho)\right).$$

Notice that $\frac{\partial g}{\partial T} = 0$ if and only if $2(1-\lambda qF_S(T-A)) + (1-\rho) = 0$, or equivalently if and only if $\rho(3-\rho) = 2\lambda qF_S(T-A)\rho$. Suppose $\frac{\partial g}{\partial T} = 0$ at some point $T_0$. Then

$$\begin{aligned} g(A,T_0) &= 1 - 2F_S(T_0-A)\lambda q\rho(A,T_0) + \rho(A,T_0) \\ &= (1-\rho(A,T_0))^2 \geq 0. \end{aligned}$$

It follows that if $g(T) < 0$ for some value $T'$ then $g(T) < 0$ for all $T > T'$. Otherwise by the mean value theorem there would exist a value $T_c$ such that $g(T_c) < 0$ and $\left.\frac{\partial g}{\partial T}\right|_{T=T_c} = 0$ which is a contradiction. Specifically if there exists a $T'$ such that $g(T') < 0$ then it must be that $\lim_{T\to\infty} g(T) < 0$. We will show that this is not possible.

Recall that $h_\infty$ is defined by the fixed point equation

$$h_\infty = 1 - (1-p)e^{-\lambda q h_\infty}.$$

Similarly we introduce the notation $\rho_\infty = (1-p)e^{-\lambda q h_\infty} = 1 - h_\infty$. Defining $g_\infty \equiv \lim_{T\to\infty} g(T)$ we have

$$\begin{aligned} g_\infty &= \lim_{T\to\infty}\left[1 - 2F_S(T-A)\lambda q\rho(A,T) + \rho(A,T)\right] \\ &= 1 - 2\lambda q\rho_\infty + \rho_\infty \end{aligned}$$

It suffices to show that $g_\infty \geq 0$. But this is exactly the result in Lemma 8, giving us our result.

$\square$

We are finally ready to establish the final Lemma.

**Lemma 10.** *For any $T \geq A$, $\lambda q > 0$ and $p \in [0,1]$, if $\frac{f_S(T-A)}{f_S(T)}$ is non-decreasing in $T$ then the function $R(A,T)$ is non-increasing in $T$.*

*Proof.* Differentiating $R(A,T)$ we obtain

$$\frac{\partial R}{\partial T} = \frac{\partial}{\partial T}\left[\frac{c}{\ell}\frac{f_S(T)}{f_S(T-A)}\frac{\rho}{1-\rho}\theta\right]$$
$$= \frac{c}{\ell}\left(\frac{\partial}{\partial T}\left[\frac{f_S(T)}{f_S(T-A)}\right]\frac{\rho}{1-\rho}\theta + \frac{f_S(T)}{f_S(T-A)}\frac{\partial}{\partial T}\left[\frac{\rho}{1-\rho}\theta\right]\right).$$

Since $\frac{f_S(T-A)}{f_S(T)}$ is non-decreasing in $T$ and $\frac{\rho}{1-\rho}\theta > 0$, it follows that $\frac{\partial}{\partial T}\left[\frac{f_S(T)}{f_S(T-A)}\right]\frac{\rho}{1-\rho}\theta \leq 0$. It suffices to show that $\frac{\partial}{\partial T}\left[\frac{\rho}{1-\rho}\theta\right] \leq 0$. Differentiating this terms gives us

$$\frac{\partial}{\partial T}\left[\frac{\rho}{1-\rho}\theta\right] = \frac{\partial}{\partial T}\left[\frac{\rho}{1-\rho}\right]\theta + \frac{\rho}{1-\rho}\frac{\partial \theta}{\partial T}.$$

Furthermore

$$\frac{\partial}{\partial T}\left[\frac{\rho}{1-\rho}\right] = -\frac{f_S(T-A)\lambda q\rho}{\theta(1-\rho)}, \tag{26}$$

and

$$\frac{\partial \theta}{\partial T} = -\frac{\lambda q f_S(T-A)\rho}{\theta}\left(1 - F_S(T-A)\lambda q\right). \tag{27}$$

Combining (26) and (27) we have

$$\frac{\partial}{\partial T}\left[\frac{\rho}{1-\rho}\theta\right] = -\frac{f_S(T-A)\lambda q\rho}{\theta(1-\rho)}\left(1 - 2F_S(T-A)\lambda q\rho + \rho\right).$$

From Lemma 9 we have $1 - 2F_S(T-A)\lambda q\rho + \rho \geq 0$. The result follows. □

We are now ready to prove the strict quasi-concavity of the Centralized Planner's expected cost function.

**Proposition 7.** *Fix $A > 0$, $\lambda q > 0$ and $p \in [0,1]$. If $\frac{f_S(T-A)}{f_S(T)}$ is non-decreasing in $T$, then the Central Planner's expected cost is strictly quasi-convex in $T$.*

*Proof.* By Lemma 10 the function $R(A,T)$ is non-increasing in $T$. Furthermore it is easy to show that the function $L(A,T)$ is monotonically increasing in $T$. It follows that the function $M(A,T)$ is either strictly positive, strictly negative or there exists a unique vale $T_0$ such that $M(A,T) < 0$ for $T < T_0$, $M(A,T_0) = 0$ and $M(A,T) > 0$ for $T > T_0$. Since $M(A,T)$ has the same sign as $\frac{\partial C}{\partial T}$ the result follows. □

## B.2 Centralized Nash Equilibrium

We are now ready to characterize the pure Nash equilibrium for the centralized botnet game on $T(\lambda)$. As we have seen the functions $L(A,T)$ and $R(A,T)$ were crucial in determining the responses for the central planner. In the proceeding analysis we will need the limiting values of these functions. First note the asymptotic behavior of the functions $\rho(A,T), V(A,T)$ and $\theta(A,T)$ as $T \downarrow A$.

$$\begin{aligned}
\lim_{T\downarrow A} \rho(A,T) &= 1-p, \\
\lim_{T\downarrow A} V(A,T) &= \frac{c}{\ell}\frac{1-p}{p}, \\
\lim_{T\downarrow A} \theta(A,T) &= 1.
\end{aligned}$$

For ease of exposition we define the following values from the asymptotic behavior of $\rho(A,T), V(A,T)$ and $\theta(A,T)$ as $T \to \infty$.

$$\begin{aligned}
\rho_\infty &\equiv \lim_{T\to\infty} \rho(A,T) = (1-p)e^{\lambda q h_\infty}, \\
V_\infty &\equiv \lim_{T\to\infty} \frac{c}{\ell}\frac{\rho(A,T)}{1-\rho(A,T)} = \frac{c}{\ell}\frac{\rho_\infty}{1-\rho_\infty}, \\
\theta_\infty &\equiv \lim_{T\to\infty} \theta(A,T) = 1-(1-p)\lambda q e^{-\lambda q h_\infty}.
\end{aligned}$$

Then for $S \sim \text{gamma}(\alpha, \beta)$ with $\alpha \geq 1$ we have the following.

$$\begin{aligned}
\lim_{T\downarrow A} R(A,T) &= \begin{cases} \exp -\beta A \frac{c}{\ell}\frac{1-p}{p} & \text{if } \alpha = 1 \\ +\infty & \text{if } \alpha > 1 \end{cases} \\
\lim_{T\downarrow A} L(A,T) &= 1 - \frac{c}{\ell}(1-F_S(A))\lambda q(1-p) \\
\lim_{T\to\infty} R(A,T) &= e^{-\beta A} V_\infty \theta_\infty, \\
\lim_{T\to\infty} L(A,T) &= 1.
\end{aligned}$$

To establish our result we will need the following lemmas which give us important properties of the best response correspondences $\sigma_b(T)$ and $\sigma_c(A)$.

**Lemma 11.** *The following properties of $\sigma_b(T)$ are true.*

1. *For $T > 0$, $0 < \sigma_b(T) < T$.*
2. *For $T > 0$, $\sigma_b(T)$ is continuously differentiable.*

3. $\limsup_{T\to\infty} \sigma_b(T) = \infty$ *with* $\limsup_{T\to\infty} (T - \sigma_b(T)) > 0$.

4. *For all* $A \in (0,\infty)$ *there exists a finite* $\tilde{T} > 0$ *such that* $\sigma_b(\tilde{T}) = A$.

*Proof.* The proof is unchanged from the decentralized case, as $\sigma_b(T)$ is independent of $\sigma_c(A)$. □

**Lemma 12.** *The following properties of* $\sigma_c(A)$ *hold.*

1. *For* $A \geq 0$, $\sigma_c(A) \geq A$.

2. *For* $0 \leq A \leq \frac{1}{\beta}\log V_\infty\theta_\infty$, $\sigma_c(A) = \infty$.

3. *For* $A > \frac{1}{\beta}\log V_\infty\theta_\infty$, $\sigma_c(A)$ *is continuously differentiable with* $\lim_{A\downarrow\frac{1}{\beta}\log V_\infty\theta_\infty} \sigma_c(A) = \infty$.

4. *Let* $A \geq \frac{1}{\beta}\log\left(\frac{c}{\ell}(1-p)(\frac{1}{p}+\lambda q)\right)$. *Then for* $\alpha = 1$ *we have* $\sigma_c(A) = A$, *and for* $\alpha > 1$ *we have* $\sigma_c(A) > A$ *with* $\lim_{A\to\infty}\sigma_c(A) - A = 0$.

*Proof.* Property 1) follows from the fact that $\frac{\partial C}{\partial T} < 0$ for $T < A$. Let $A \leq \frac{1}{\beta}\log V_\infty\theta_\infty$. Then $\lim_{T\to\infty} R(A,T) = \frac{c}{\ell}e^{-\beta A}\frac{\rho_\infty}{1-\rho_\infty}\theta_\infty \leq 1 = L_\infty < L(A,T)$ for all $T \in [A,\infty)$. Thus $\frac{\partial C}{\partial T} \leq 0$ which implies $\phi_p(A) = \infty$, establishing property 2).

Now let $A > \frac{1}{\beta}\log V_\infty\theta_\infty$. That $\sigma_c(A)$ is single-valued follows from the strict quasi-concavity of $C(A,T)$ in $T$. For any $(a,t) \in \mathbb{R}^2$ define

$$g(a,t) = R(a,t) - L(a,t).$$

Setting $g(a,t) = 0$ and applying the implicit function theorem gives us the existence of a continuously differentiable function $v(a)$ such that $g(a,v(a)) = 0$ for all $a$ in some open neighborhood of $A$. Since $\sigma_p(A)$ is the unique value satisfying $g(A,\sigma_p(A)) = 0$ for all $A > \frac{1}{\beta}\log V_\infty$, we must have $v(A) = \sigma_p(A)$ for all $A > \frac{1}{\beta}\log V_\infty$. Thus $\sigma_p(A)$ is continuously differentiable for all $A > \frac{1}{\beta}\log V_\infty$.

Denote $A_\infty = \frac{1}{\beta}\log V_\infty\theta_\infty$. To show $\lim_{A\downarrow A_\infty}\sigma_c(A) = \infty$ it suffices to show that for any $M \in \mathbb{R}^+$ there exists an $\epsilon > 0$ such that $\sigma_c(A_\infty+\epsilon) > M$. By monotonicity and continuity of $R(A,T)$ and $L(A,T)$ in $T$, it suffices to show that for any $M \in \mathbb{R}^+$ there exists an $\epsilon > 0$ and $\delta > 0$ (which may depend on $\epsilon$) such that $R(A_\infty+\epsilon, M) > L(A_\infty+\epsilon, M)$ and $R(A_\infty+\epsilon, M+\delta) < L(A_\infty+\epsilon, M+\delta)$.

Fix $M > A_\infty$ and choose $\epsilon < \min\left\{M - A_\infty, \frac{1}{\beta}\log\left(\left(\frac{M}{M-A_\infty}\right)^{\alpha-1}\frac{V(A_\infty,M)\theta(A_\infty,M)}{V_\infty\theta_\infty}\right)\right\}$. By Lemma 10 $\frac{f_S(T)}{f_S(T-A_\infty)}V(A_\infty,T)\theta(A_\infty,T)$ is non-increasing in $T$. This gives

$$\frac{R(A_\infty,M)}{\lim_{T\to\infty} R(A_\infty,T)} > 1.$$

But $\lim_{T\to\infty} R(A_\infty, T) = 1$ which implies $\frac{V(A_\infty,M)\theta(A_\infty,M)}{V_\infty\theta_\infty} > 1$. Notice this guarantees $\epsilon > 0$. Furthermore since $V(A,T), \theta(A,T)$ and $\rho(A,T)$ depend on $A$ and $T$ only through the difference $T - A$ we have

$$\begin{aligned} V(A_\infty + \epsilon, M) &= V(A_\infty, M - \epsilon) \\ \theta(A_\infty + \epsilon, M) &= \theta(A_\infty, M - \epsilon) \\ \rho(A_\infty + \epsilon, M) &= \rho(A_\infty, M - \epsilon). \end{aligned}$$

We now have

$$\begin{aligned} L(A_\infty + \epsilon, M) \leq 1 &< e^{-\beta\epsilon}\left(\frac{M}{M - A_\infty}\right)^{\alpha-1}\frac{V(A_\infty, M)\theta(A_\infty, M)}{V_\infty\theta_\infty} \\ &= e^{-\beta\epsilon}e^{-\beta A_\infty}\left(\frac{M}{M - A_\infty}\right)^{\alpha-1} V(A_\infty, M)\theta(A_\infty, M) \\ &\leq e^{-\beta(A_\infty+\epsilon)}\left(\frac{M}{M - \epsilon - A_\infty}\right)^{\alpha-1} V(A_\infty, M-\epsilon)\theta(A_\infty, M-\epsilon) \\ &= e^{-\beta(A_\infty+\epsilon)}\left(\frac{M}{M - (A_\infty + \epsilon)}\right)^{\alpha-1} V(A_\infty + \epsilon, M)\theta(A_\infty + \epsilon, M) \\ &= R(A_\infty + \epsilon, M). \end{aligned}$$

Now choose $\delta_1 > 0$ such that

$$\left(\frac{M + \delta_1}{M + \delta_1 - \epsilon - A_\infty}\right)^{\alpha-1} V(A_\infty, M + \delta_1 - \epsilon)\theta(A_\infty, M + \delta_1 - \epsilon) < e^{\beta\epsilon}V_\infty\theta_\infty$$

and $\delta_2 > 0$ such that

$$e^{-\beta(M+\delta_2)} < \frac{\ell}{c\lambda q(1-p)}\left(1 - e^{-\beta\epsilon}\left(\frac{M + \delta_1}{M + \delta_1 - \epsilon - A_\infty}\right)^{\alpha-1}\frac{V(A_\infty, M + \delta_1 - \epsilon)\theta(A_\infty, M + \delta_1 - \epsilon)}{V_\infty\theta_\infty}\right).$$

Both $\delta_1$ and $\delta_2$ are guaranteed to exist by the continuity and monotonicity of $R(A,T)$ and $\rho(A,T)$ in $T$. Setting $\delta = \max\{\delta_1, \delta_2\}$ we have

$$e^{-\beta\epsilon}\left(\frac{M + \delta}{M + \delta - \epsilon - A_\infty}\right)^{\alpha-1}\frac{V(A_\infty, M + \delta - \epsilon)\theta(A_\infty, M + \delta - \epsilon)}{V_\infty\theta_\infty} + \frac{c}{\ell}e^{-\beta(M+\delta)}\lambda q\rho(A_\infty, M + \delta - \epsilon) < 1,$$

which is equivalent to $R(A_\infty + \epsilon, M + \delta) < L(A_\infty + \epsilon, M + \delta)$.

Now suppose $A \geq \frac{1}{\beta}\log\left(\frac{c}{\ell}(1-p)(\frac{1}{p} + \lambda q)\right)$. For $\alpha = 1$ it follows that $e^{-\beta A}\frac{c}{\ell}\frac{1-p}{p} \leq 1 - \frac{c}{\ell}e^{-\beta A}\lambda q(1-p)$, i.e.

$$\lim_{T\downarrow A} L(A,T) \geq \lim_{T\downarrow A} R(A,T).$$

By monotonicity $L(A,T) \geq R(A,T)$ and hence $\frac{\partial C}{\partial T} \geq 0$ for all $T \geq A$. It follows that $\sigma_c(A) = A$.

For $\alpha > 1$ we need to show $\lim_{A\to\infty} \sigma_c(A) - A = 0$. Define $m = \max_{A,T\in\mathbb{R}^+} \{V(A,T)\theta(A,T)\}$. We are guaranteed $0 < m < \infty$ by the boundedness of $V(A,T)$ and $\theta(A,T)$. Fix $\epsilon > 0$. Since

$$\lim_{A\to\infty} \frac{A}{[m^{-1}e^{\beta A}(1 - \frac{c}{\ell}(1 - F_S(A))\lambda q(1-p))]^{\frac{1}{\alpha-1}} - 1} = +\infty$$

we can find an $A_0$ such that

$$\frac{A_0}{[m^{-1}e^{\beta A_0}(1 - \frac{c}{\ell}(1 - F_S(A_0))\lambda q(1-p))]^{\frac{1}{\alpha-1}} - 1} < \epsilon.$$

It follows that

$$\left(\frac{A_0}{\epsilon} + 1\right)^{\alpha-1} V(A_0, A_0+\epsilon)\theta(A_0, A_0+\epsilon) \leq \left(\frac{A_0}{\epsilon} + 1\right)^{\alpha-1} m < e^{\beta A_0}\left(1 - \frac{c}{\ell}(1 - F_S(A_0))\lambda q(1-p)\right).$$

This gives

$$\left(\frac{A_0+\epsilon}{A_0+\epsilon-A_0}\right)^{\alpha-1} e^{-\beta A_0} V(A_0, A_0+\epsilon)\theta(A_0, A_0+\epsilon) < 1 - \frac{c}{\ell}(1 - F_S(A_0))\lambda q \rho(A_0, A_0),$$

i.e. for all $T \geq A_0$

$$L(A_0, T) \geq L(A_0, A_0) > R(A_0, A_0+\epsilon).$$

Note that as $\lim_{\epsilon\downarrow 0} R(A_0, A_0+\epsilon) = +\infty$, thus there exists an $\epsilon_1 < \epsilon$ such that $R(A_0, A_0 + \epsilon_1) > 1 \geq L(A_0, T)$ for all $T \geq A_0$. By the continuity of $R$ and $L$ there exists an $\epsilon_2 \in (\epsilon_1, \epsilon)$ such that $L(A_0, A_0+\epsilon_2) = R(A_0, A_0+\epsilon_2)$. It follows that $\sigma_c(A_0) = A_0 + \epsilon_2$. Hence

$$0 < \sigma(A_0) - A_0 = \epsilon_2 < \epsilon.$$

Taking $\epsilon \to 0$ gives us the result.

□

Using the previous technical lemmas we can establish the following lemma which will help us prove the existence and uniqueness of a Nash equilibrium.

**Lemma 13.** *For any* $(A,T) \in \mathbb{R}^2$ *with* $\frac{1}{\beta}\log V_\infty\theta_\infty < A < T$ *we have*

$$\frac{d}{dA}\left[\sigma_c(A) - A\right] < 0 < \frac{d}{dT}\left[T - \sigma_b(T)\right].$$

*Or equivalently*

$$\frac{d\sigma_c}{dA} < 1,$$
$$\frac{d\sigma_b}{dT} < 1.$$

*Proof.* The proof that $0 < \frac{d}{dT}\left[T - \sigma_b(T)\right]$ is exactly as in the decentralized case. The proof for $\sigma_c(A)$ is analogous to the decentralized case.

Recall the definitions for $R(A,T)$ and $L(A,T)$. For each $A > \frac{1}{\beta}\log V_\infty \theta_\infty$ the function $\sigma_c(A)$ is defined as the unique solution $T > A$ to the equation

$$L(A,T) = R(A,T)$$

Thus we have for each $A > \frac{1}{\beta}\log V_\infty \theta_\infty$ the relation

$$L(A,\sigma_c(A)) = R(A,\sigma_c(A))$$

For ease of exposition define the following functions.

$$\begin{aligned} m(A) &= \lambda^2 q^2 [1 - F_S(\sigma_c(A))] \frac{f_S(\sigma_c(A) - A)\rho(\sigma_c(A) - A)(1 - \rho(\sigma_c(A) - A))}{\theta(\sigma_c(A) - A)} \\ w(A) &= \frac{d}{dx}\left[\frac{f_S(x)}{f_S(x - A)} \frac{\rho(x - A)}{1 - \rho(x - A)} \theta(x - A)\right]_{x = \sigma_c(A)} \end{aligned}$$

For finite $A > \frac{1}{\beta}\log V_\infty \theta_\infty$ we have $m(A) > 0$ and from Lemma 10 we have $w(A) \leq 0$. Thus $m(A) - w(A) > 0$. By the implicit function theorem and chain rule we can then write

$$\frac{d\sigma_c}{dA} = 1 + \frac{f_S(\sigma_p(A) - A) f_S'(\sigma_p(A))}{[m(A) - w(A)] f_S(\sigma_p(A) - A)^2}. \tag{28}$$

Clearly $\frac{d\sigma_p}{dA} < 1$ if and only if $f_S'(\sigma_p(A)) < 0$. If $S \sim \text{gamma}(\alpha, \beta)$ then this is equivalent to $\frac{d\sigma_p}{dA} < 1$ if and only if $\sigma_p(A) > \frac{\alpha - 1}{\beta}$. For $\alpha = 1$ we are done. Consider the case $\alpha > 1$. Suppose there exists an $\tilde{A}$ such that $\sigma_p(A) \leq \frac{\alpha - 1}{\beta}$. By the properties of $\sigma_p(A)$ from Lemma 4 there must exist a global minimum at some point $A_0$ such that $\left.\frac{d\sigma_p}{dA}\right|_{A = A_0} = 0$ and $\sigma_p(A_0) \leq \sigma_p(\tilde{A})$. Thus $\sigma_p(A_0) \leq \frac{\alpha - 1}{\beta}$ which implies $\left.\frac{d\sigma_p}{dA}\right|_{A = A_0} \geq 1$ which is a contradiction. It follows that $\sigma_p(A) > \frac{\alpha - 1}{\beta}$ from which we obtain $\frac{d\sigma_p}{dA} < 1$.

We immediately obtain the desired result, namely

$$\frac{d}{dA}\left[\sigma_c(A) - A\right] < 0 < \frac{d}{dT}\left[T - \sigma_b(T)\right].$$

□

We are now ready to prove the existence and uniqueness of a pure Nash equilibrium in the centralized botnet game.

**Theorem 2.** *Let $S \sim Gamma(\alpha, \beta)$ with $\alpha \geq 1, \beta > 0$ in the centrally planned, infinite-population, botnet game on $T(\lambda)$ with homogeneous agents. Then there exists a unique, pure, symmetric Nash equilibrium in which all agents play the same strategy, i.e. there exists a unique point $(A^*, T^*) \in \mathbb{R}^2$ such that*

$$\sigma_b(T^*) = A^*$$
$$\sigma_c(A^*) = T^*.$$

*Proof.* Given Lemma's 11, 12 and 13 the proof is analogous to the decentralized case. □

Again we can obtain bounds on the equilibrium.

**Corollary 5.** *Let $(A^*, T^*)$ be the unique, pure Nash equilibrium in Theorem 2. Then we have $T^* \geq \max\{A^*, \frac{\alpha-1}{\beta}\}$ and $A^* \geq \max\left\{0, \frac{1}{\beta}\log(V_\infty\theta_\infty)\right\}$. In the special case $\alpha = 1$, i.e. $S_i \sim exp(\beta)$, we have*

$$\max\left\{0, \frac{1}{\beta}\log(V_\infty\theta_\infty)\right\} \leq A^* \leq \max\left\{0, \frac{1}{\beta}\log\left(\frac{c}{\ell}(1-p)\left(\frac{1}{p} + \lambda q\right)\right)\right\},$$
$$A^* \leq T^* < \inf\left\{T : \sigma_b(T) = \frac{c}{\ell}(1-p)\left(\frac{1}{p} + \lambda q\right)\right\}$$

# C Extension of Equilibrium Results to $G(n, \lambda/n)$

## C.1 Convergence Results for the Centralized Botnet Game

The preceding analysis is applicable to the limiting object of a sequence of random rooted Poisson Branching Process $T_n(\lambda) \to T(\lambda)$. In this section we show that Nash equilibria on $T(\lambda)$ are also Nash equilibria in the same game played on the limiting graph of a sequence of Erdos-Renyi random graphs $G(n, \lambda/n)$, which we denote by $G_\infty(\lambda)$. The proof relies on the objective method [2] and follows the proof in [11].

Notice that for a given $A$ and $T$ an agent's cost and the botmaster's utility are random variables. Fixing $A \in \mathcal{A}$ and $T \in \mathcal{T}$ let $C_i^{(n)}(A, T)$ be the random cost of agent $i$, ($i = 1, 2, ..., n$) and $U_b^{(n)}(A, T)$ the random utility of the bot master on $G(n, \lambda/n)$. Let $X_i^{(n)}(A, T)$ be the indicator random variable for a false alarm and $Y_i^{(n)}(A, T)$ be the indicator random variable for a missed detection for agent $i$ on $G(n, \lambda/n)$. Furthermore let $W_i^{(n)}(A, T)$ be the indicator random variable for infection of agent $i$ on $G(n, \lambda/n)$ and let $D_i^{(n)}(A, T)$ be the indicator random variable for a detection event by agent $i$ on $G(n, \lambda/n)$. If agent $i$ and agent $j$ are neighbors in $G(n, \lambda/n)$ then we write $i \sim j$. We will suppress the $A, T$

dependence notation from here on. With the above notation we have the following relations.

$$W_i^{(n)} = 1 - (1 - \chi_i^{(n)}) \prod_{i \sim j} (1 - B_{ki}^{(n)} Y_i^{(n)}) \tag{29}$$

$$D_i^{(n)} = \mathbb{1}_{\{T < W_i^{(n)} + S_i^{(n)} A\}} \tag{30}$$

$$X_i^{(n)} = (1 - W_i^{(n)}) D_i^{(n)} \tag{31}$$

$$Y_i^{(n)} = W_i^{(n)} (1 - D_i^{(n)}) \tag{32}$$

Let

$$\begin{aligned}
C_i^{(n)} &= cX_i^{(n)} + \ell Y_i^{(n)}, \\
C^{(n)} &= \frac{1}{n} \sum_{i=1}^{n} C_i^{(n)} = c\frac{1}{n} \sum_{i=1}^{n} X_i^{(n)} + \ell \frac{1}{n} \sum_{i=1}^{n} Y_i^{(n)} \\
U_b^{(n)} &= A \frac{1}{n} \sum_{i=1}^{n} Y_i^{(n)}.
\end{aligned}$$

The expected cost and utilities are then

$$\begin{aligned}
E[C_i^{(n)}] &= cE[X_i^{(n)}] + \ell E[Y_i^{(n)}], \\
E[C^{(n)}] &= c\frac{1}{n} \sum_{i=1}^{n} E[X_i^{(n)}] + \ell \frac{1}{n} \sum_{i=1}^{n} E[Y_i^{(n)}] \\
E[U_b^{(n)}] &= A \frac{1}{n} \sum_{i=1}^{n} E[Y_i^{(n)}].
\end{aligned}$$

Because the underlying graph $G(n, \lambda/n)$ is random the labeling of nodes is interchangeable and by exchangeability we have for all $i \neq j$

$$\begin{aligned}
E[X_i^{(n)}] &= E[X_j^{(n)}], \\
E[Y_i^{(n)}] &= E[Y_j^{(n)}].
\end{aligned}$$

In particular the root node of $G(n, \lambda/n)$, say node $i = 0$ is chosen uniformly at random, thus we have for all $i = 0, 1, 2, 3..., n - 1$

$$E[C_i^{(n)}] = cE[X_0^{(n)}] + \ell E[Y_0^{(n)}], \tag{33}$$

$$E[C^{(n)}] = c\frac{1}{n} \sum_{i=1}^{n} E[X_0^{(n)}] + \ell \frac{1}{n} \sum_{i=1}^{n} E[Y_0^{(n)}] = cE[X_0^{(n)}] + \ell E[Y_0^{(n)}] \tag{34}$$

$$E[U_b^{(n)}] = A \frac{1}{n} \sum_{i=1}^{n} E[Y_0^{(n)}] = AE[Y_0^{(n)}]. \tag{35}$$

**Proposition 8.** *For any $(A,T) \in \mathcal{A}\times\mathcal{T}$ if the processes $\left\{X_i^{(n)}(A,T)\right\}_{i=0}^{n-1}$ and $\left\{Y_i^{(n)}(A,T)\right\}_{i=0}^{n-1}$ satisfy (29) - (32) on $G(n,\lambda/n)$, then*

$$\lim_{n\to\infty} E\left[X_i^{(n)}(A,T)\right] = [1-F_S(T)](1-p)e^{-\lambda q h(A,T)}$$
$$\lim_{n\to\infty} E\left[Y_i^{(n)}(A,T)\right] = h(A,T).$$

*Proof.* For $d>0$ let $N_d(1,G(n,\lambda/n))$ be a neighborhood of radius $d$ about the root node $i=1$ of $G(n,\lambda/n)$. For fixed $d$ we have $G(n,\lambda/n) \xrightarrow{\mathcal{D}} T(\lambda,d)$ as $n\to\infty$. By the Skorohod Representation Theorem we can consider the two random graphs to be defined on the same probability space and with probability one, there is a finite random variable $N$ such that $N_d(0,G(n,\lambda/n)) = T(\lambda,d)$ for all $n \geq N$. Fix $d>0$ and denote the leaves of $T(\lambda,d)$ by $\partial T(\lambda,d)$. We now construct two depth-$d$ recursive tree processes, $L_i^{(d)}$ and $U_i^{(d)}$. For $i \in \partial T(\lambda,d)$ let

$$L_i^{(d)} = \chi_i \mathbb{1}(T \leq S_i + \chi_i A)$$
$$U_i^{(d)} = 1.$$

For any recursive tree process (RTP) $R_i$ defined for each $i \in T(\lambda)$ define the functionals $W(\cdot)$ and $D(\cdot)$ as follows.

$$W(R_i) = 1-(1-\chi_i)\prod_{j\to i}\left(1-B_{ji}R_j\right)$$
$$D(R_i) = \mathbb{1}_{\left\{T<S_i+W\left(R_i\right)A\right\}}$$

Thus the functional $W(\cdot)$ and $D(\cdot)$ are actually functionals of all children of the argument $R_i$. For all $i \notin \partial T(\lambda,d)$ we define

$$L_i^{(d)} = W\left(L_i^{(n)}\right) D\left(L_i^{(n)}\right)$$
$$U_i^{(d)} = W\left(U_i^{(n)}\right) D\left(U_i^{(n)}\right).$$

For $n>N$ we can consider $N_d(0,G(n,\lambda/n)) = T(\lambda)$. We can then define the corresponding RTP $\left\{\tilde{Y}_i^{(n)}(A,T)\right\}_{i=0}^{n-1}$ for $n>N$ by

$$\tilde{Y}_i^{(n)}(A,T) = \begin{cases} Y_i^{(n)}(A,T) & \text{if } i \in \partial T(\lambda,d) \\ W\left(\tilde{Y}_i^{(n)}(A,T)\right) D\left(\tilde{Y}_i^{(n)}(A,T)\right) & \text{if } i \notin \partial T(\lambda,d). \end{cases}$$

Observe that for $n > N$ we have $E[Y_0^{(n)}] = E[\tilde{Y}_0^{(n)}]$. This is not necessarily true for $i \neq 0$, but we are only concerned about the root here.

First observe that for all $n \geq N$ and for all $i \in \partial T(\lambda, d)$ we have $L_i^{(d)} \leq \tilde{Y}_i^{(n)} \leq U_i^{(d)}$. We will show that in fact $L_i^{(d)} \leq \tilde{Y}_i^{(n)} \leq U_i^{(d)}$ holds for all $i$ of equal depth in the tree, in particular

$$L_{\o}^{(d)} \leq \tilde{Y}_0^{(n)} \leq U_{\o}^{(d)}. \tag{36}$$

We prove (36) by showing that the functionals $W(\cdot)$ and $D(\cdot)$ are monotonic, i.e. for any indicator random variables $Q_i, R_i$ defined for each $i \in T(\lambda)$, if $Q_j \leq R_j$ for each $j$ such that $j \to i$, then $W(Q_i) \leq W(R_i)$ and $D(Q_i) \leq D(R_i)$. To prove this we consider the different cases. First note that if $\chi_i = 1$ then $W(Q_i) = W(R_i) = 1$. Suppose $\chi_i = 0$. If $\prod_{j\to i}(1 - B_{ji}Q_j) = \prod_{j\to i}(1 - B_{ji}R_j)$ then $W(Q_i) = W(R_i) = 1$. Suppose $\prod_{j\to i}(1 - B_{ji}Q_j) \neq \prod_{j\to i}(1 - B_{ji}R_j)$. Then there are two possibilities. Either

$$0 = \prod_{j\to i}(1 - B_{ji}Q_j) < \prod_{j\to i}(1 - B_{ji}R_j) = 1 \tag{37}$$

or

$$1 = \prod_{j\to i}(1 - B_{ji}Q_j) > \prod_{j\to i}(1 - B_{ji}R_j) = 0. \tag{38}$$

Suppose (37) is true. Then $B_{ji}R_j = 0$ for all $j$ such that $(i, j) \in E$ while at the same time $B_{ji}Q_j = 1$ for some $j$ such that $(i, j) \in E$. Let $j^*$ be such that $B_{j^*i}Q_{j^*} = 1$. Then we must have $B_{j^*i} = Q_{j^*} = 1$. But then $R_{j^*} = 0$ giving us $R_{j^*} < Q_{j^*}$. This contradicts our assumption that $Q_j \leq R_j$. It follows that (38) must hold, which implies $0 = W(Q_j) < W(R_j) = 1$. This exhausts all possibilities.

The proof for the monotonicity of the functional $D(\cdot)$ follows directly from the monotonicity of $W(\cdot)$. Specifically if $W(Q_i) \leq W(R_i)$ then we need only consider the two cases. If $W(Q_i) = W(R_i)$ then $D(Q_i) = D(R_i)$. If $W(Q_i) \neq W(R_i)$ then we have $W(Q_i) = 0$ and $W(R_i) = 1$, in which case $D(Q_i) = \mathbb{1}_{\{T_i < S_i\}}$ and $D(R_i) = \mathbb{1}_{\{T_i < S_i + A\}}$. If $T_i < S_i$ then $D(Q_i) = D(R_i) = 1$. If $S_i \leq T_i < S_i + A$ then $0 = D(Q_i) < D(R_i) = 1$. Finally if $T_i \geq S_i + A$ then $D(Q_i) = D(R_i) = 0$. Hence $D(Q_i) \leq D(R_i)$.

By the monotonicity of both $W(\cdot)$ and $D(\cdot)$ we have the monotonicity of $W(\cdot)D(\cdot)$. Thus for all $i$ at depth $d-1$ from the root we must have $L_i^{(d)} \leq \tilde{Y}_i^{(n)} \leq U_i^{(d)}$. If $d = 1$ we then have (36) trivially. By induction on $d$ we obtain the result for any finite $d$.

We now have

$$E[L_{\o}^{(d)}] \leq E[\tilde{Y}_0^{(n)}] = E[Y_0^{(n)}] \leq E[U_{\o}^{(d)}]. \tag{39}$$

In order to finish the proof we show that $L_{\o}^{(d)}$ and $U_{\o}^{(d)}$ both converge in distribution to Bernoulli random variables with parameter $h(A, T)$ as $d \to \infty$.

Define $h_i^{(d)} = P\left(L_i^{(d)} = 1\right)$. For $d = 1$ we have

$$L_\emptyset^{(1)} = 1 - \max\left\{1 - W\left(L_\emptyset^{(1)}\right), \mathbb{1}\{T_\emptyset \leq S_\emptyset + W\left(L_\emptyset^{(1)}\right) A\}\right\},$$
$$W\left(L_\emptyset^{(1)}\right) = 1 - (1 - \chi_i) \prod_{j \to i} \left(1 - B_{ji} L_j^{(1)}\right).$$

By definition for $j \in \partial T(\lambda, 1)$

$$\begin{aligned} h_j^{(1)} &= P\left(L_j^{(1)} = 1\right) \\ &= P\left(\chi_j = 1, T_j > S_j + \chi_j A\right) \\ &= F_S(T_j - A)p. \end{aligned}$$

It is then possible to show by a similar derivation as we did to get $h(A, T)$ that

$$\begin{aligned} h_\emptyset^{(1)} &= F_S(T_\emptyset - A)[1 - (1-p)e^{-\lambda q h_1^{(1)}}], \\ &= F_S(T_\emptyset - A)[1 - (1-p)e^{-\lambda q F_S(T_\emptyset - A)p}]. \end{aligned}$$

Define the function $g(x, A, T) = F_S(T - A)[1 - (1-p)e^{-\lambda q x}]$. The above gives $h_\emptyset^{(1)} = g(h_1^{(1)}, A, T)$. By induction on $d$ it is straight forward to show that $h_\emptyset^{(d+1)} = g(h_1^{(d+1)}, A, T) = g^d(F_S(T - A)p, A, T)$ where superscript $d$ represents composition in $x$. Thus as $d \to \infty$ repeated composition of the function $g(\cdot, A, T)$ will converge to the unique fixed point solution $h(A, T)$. The proof for $U_\emptyset^{(d)}$ is analogous.

With the above we have $\lim_{d\to\infty} E[L_\emptyset^{(d)}] = h(A, T)$ and $\lim_{d\to\infty} E[U_\emptyset^{(d)}] = h(A, T)$. Then in the limit as $d \to \infty$ we must have $E[Y_0^{(n)}] = h(A, T)$ for $n \geq N$. With these results a similar argument shows that in the limit as $d \to \infty$ for $n \geq N$ we must have

$$E[X_0^{(n)}] = [1 - F_S(T_\emptyset)](1-p)e^{-\lambda q h(A,T)}.$$

□

Applying the above proposition to (34) and (35) we get the following corollary.

**Corollary 6.** *For any $(A, T) \in \mathcal{A} \times \mathcal{T}$ and corresponding processes $\{X_i^{(n)}(A, T)\}_{i=0}^{n-1}$ and $\{Y_i^{(n)}(A, T)\}_{i=0}^{n-1}$ satisfying (29) - (32) on $G(n, \lambda/n)$ we have*

$$\lim_{n\to\infty} E[C^{(n)}(A, T)] = c[1 - F_S(T)](1-p)e^{-\lambda q h(A,T)} + \ell h(A, T) = C(A, T),$$
$$\lim_{n\to\infty} E[U_b^{(n)}(A, T)] = A h(A, T) = U(A, T).$$

Given the above proposition and corollary we have the following.

**Proposition 9.** *Any pure, symmetric Nash equilibrium $(A^*, T^*)$ in the centralized botnet game on $T(\lambda)$ is a pure, symmetric Nash equilibrium in the centralized botnet game on $G^\infty(\lambda)$.*

## C.2 Convergence Results for the Decentralized Botnet Game

Now consider the case for the decentralized game. We still work in the same probability space but our strategy space is now $\mathcal{A} \times \mathcal{T} \times \mathcal{T}$.

As before for a given $(A, T, T_\emptyset) \in \mathcal{A} \times \mathcal{T} \times \mathcal{T}$ a root agent's cost and the botmaster's utility are random variables. Let $C_0^{(n)}(A, T, T_\emptyset)$ be the random cost of a root agent and $U_b^{(n)}(A, T)$ the random utility of the bot master on $G(n, \lambda/n)$. As before let $X_i^{(n)}(A, T)$ be the indicator random variable for a false alarm and $Y_i^{(n)}(A, T)$ be the indicator random variable for a missed detection for agent $i > 0$ on $G(n, \lambda/n)$ and denote by $X_0^{(n)}(A, T, T_\emptyset)$ and $Y_0^{(n)}(A, T, T_\emptyset)$ the indicator random variables for false alarm and missed detection, respectively, for a root agent. The defining relations analogous to (29)-(32) are as follows.

$$W_i^{(n)} = 1 - (1 - \chi_i^{(n)}) \prod_{i \sim j} (1 - B_{ki}^{(n)} Y_i^{(n)}) \tag{40}$$

$$D_i^{(n)} = \begin{cases} \mathbb{1}_{\{T_\emptyset < W_0^{(n)} + S_i^{(n)} A\}} & \text{if } i = 0 \\ \mathbb{1}_{\{T < W_i^{(n)} + S_i^{(n)} A\}} & \text{if } i > 0 \end{cases} \tag{41}$$

$$X_i^{(n)} = (1 - W_i^{(n)}) D_i^{(n)} \tag{42}$$

$$Y_i^{(n)} = W_i^{(n)} (1 - D_i^{(n)}) \tag{43}$$

The random cost to the root agent and the random utility to the bot master are then

$$\begin{aligned} C_0^{(n)}(A, T, T_\emptyset) &= c X_0^{(n)}(A, T, T_\emptyset) + \ell Y_0^{(n)}(A, T, T_\emptyset), \\ U_b^{(n)}(A, T, T_\emptyset) &= A \frac{1}{n} \sum_{i=0}^{n-1} Y_i^{(n)} = A \frac{1}{n} Y_0^{(n)}(A, T, T_\emptyset) + A \frac{1}{n} \sum_{i=1}^{n-1} Y_i^{(n)}(A, T). \end{aligned}$$

The expected cost and utilities become

$$E[C_0^{(n)}(A, T, T_\emptyset)] = c E[X_0^{(n)}(A, T, T_\emptyset)] + \ell E[Y_0^{(n)}(A, T, T_\emptyset)], \tag{44}$$

$$E[U_b^{(n)}(A, T, T_\emptyset)] = A \frac{1}{n} E[Y_0^{(n)}(A, T, T_\emptyset)] + A \frac{1}{n} \sum_{i=1}^{n-1} E[Y_i^{(n)}(A, T)]. \tag{45}$$

Since a root node is chosen uniformly at random we have by exchangeability for all $i, j \neq 0$

$$\begin{aligned} E[X_i^{(n)}] &= E[X_j^{(n)}], \\ E[Y_i^{(n)}] &= E[Y_j^{(n)}]. \end{aligned}$$

Thus we can write $E[U_b^{(n)}(A, T, T_\emptyset)] = A \frac{1}{n} E[Y_0^{(n)}(A, T, T_\emptyset)] + A \frac{n-1}{n} E[Y_1^{(n)}(A, T)]$. Then $\lim_{n\to\infty} E[U_b^{(n)}(A, T, T_\emptyset)] = \lim_{n\to\infty} A E[Y_1^{(n)}(A, T)]$ provided this limit exists. Thus we can consider the limiting expected utility of the bot master as a function of $A$ and $T$ only. In addition if we can show that $\lim_{n\to\infty} E[C_0^{(n)}(A,T,T_\varnothing)] = C_\varnothing(A,T,T_\varnothing)$, then by our previous equilibrium results there will exist an optimal population strategy $T^*(A)$. In this case all agents will play the same strategy, i.e. $T = T_\varnothing = T^*(A)$ and by exchangeability we will have for all $i \neq j$

$$\begin{aligned} E[X_i^{(n)}] &= E[X_j^{(n)}], \\ E[Y_i^{(n)}] &= E[Y_j^{(n)}]. \end{aligned}$$

In particular we have

$$E[U_b^{(n)}(A,T)] = A\frac{1}{n}\sum_{i=0}^{n-1} E[Y_0^{(n)}] = AE[Y_0^{(n)}].$$

Thus it suffices to prove the that

$$\begin{aligned} \lim_{n\to\infty} E[X_0^{(n)}(A,T,T_\varnothing)] &= E[X_\varnothing(A,T,T_\varnothing)], \\ \lim_{n\to\infty} E[Y_0^{(n)}(A,T,T_\varnothing)] &= E[Y_\varnothing(A,T,T_\varnothing)]. \end{aligned}$$

The proof of this convergence is exactly as in the centralized case. Thus we state the corresponding propositions for the decentralized game without proof.

**Proposition 10.** *For any $(A,T,T_\varnothing) \in \mathcal{A}\times\mathcal{T}\times\mathcal{T}$ if the processes $\left\{X_i^{(n)}\right\}_{i=0}^{n-1}$ and $\left\{Y_i^{(n)}\right\}_{i=0}^{n-1}$ satisfy (40) - (43) on $G(n,\lambda/n)$, then*

$$\begin{aligned} \lim_{n\to\infty} E\left[X_0^{(n)}\right] &= [1-F_S(T_\varnothing)](1-p)e^{-\lambda q h(A,T)}, \\ \lim_{n\to\infty} E\left[Y_0^{(n)}\right] &= F_S(T_\varnothing - A)[1-(1-p)e^{-\lambda q h(A,T)}]. \end{aligned}$$

Applying the above proposition to (44) and (45) we get the following corollary.

**Corollary 7.** *For any $(A,T,T_\varnothing) \in \mathcal{A}\times\mathcal{T}\times\mathcal{T}$ and corresponding processes $\{X_i^{(n)}(A,T)\}_{i=0}^{n-1}$ and $\{Y_i^{(n)}(A,T)\}_{i=0}^{n-1}$ satisfying (40) - (43) on $G(n,\lambda/n)$ we have*

$$\begin{aligned} \lim_{n\to\infty} E[C^{(n)}(A,T)] &= c[1-F_S(T)](1-p)e^{-\lambda q h(A,T)} + \ell F_S(T_\varnothing - A)[1-(1-p)e^{-\lambda q h(A,T)}] \\ &= C_\varnothing(A,T,T_\varnothing), \\ \lim_{n\to\infty} E[U_b^{(n)}(A,T)] &= Ah(A,T) \\ &= U(A,T). \end{aligned}$$

Given the above proposition and corollary we have the following.

**Proposition 11.** *Any pure, symmetric Nash equilibrium* $(A^*, T^*)$ *in the decentralized botnet game on* $T(\lambda)$ *is a pure, symmetric Nash equilibrium in the decentralized botnet game on* $G^\infty(\lambda)$.

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