# Turn-Based Qualitative Solution of Concurrent Parity Games*

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**Abstract**

We consider two-player concurrent games played on graphs, where at each state both players choose moves simultaneously and independently. We consider $\omega$-regular winning conditions specified as parity objectives and study the qualitative winning mode, i.e., whether a player can win with probability arbitrarily close to 1 (*limit-winning*). We provide an efficient reduction from limit-winning concurrent parity games to winning turn-based parity games, where at each state only one player has a choice of moves. From a theoretical point of view, the reduction shows that for the qualitative winning mode, one can eliminate the concurrent nature of a game, and since turn-based games are well-studied, the reduction improves the understanding of concurrent games. From a practical point of view, the reduction provides algorithms for limit-winning concurrent parity games from algorithms for solving turn-based parity games. Every improvement in the latter algorithms will carry over to the former. In particular, using recent results on algorithms for turn-based parity games, we improve the best known time complexity to solve concurrent parity games when the set of available moves at each state is small.

## 1 Introduction

Two-player games played on graphs with $\omega$-regular winning conditions have deep theoretical connections to logic and automata [17, 15, 10, 8, 21] as well as many applications in the synthesis and verification of reactive systems [19, 16, 1, 6, 2]. These games come in two varieties. The simple variety is referred to as *turn-based* games. In turn-based games, the vertices of a graph are partitioned into player-1 states and player-2 states. If the vertex of a game is a player-1 state, then player 1 chooses an outgoing edge, whose destination is the next state of the game; and symmetrically for player-2 states. The result of an infinite game is an infinite path through the graph, and

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player 1 wins iff that path lies within the winning set. We will focus on *parity* winning conditions, which is a common and complete way of specifying $\omega$-regular sets of winning paths. The algorithmics of turn-based games in general —and of turn-based parity games in particular— has been a topic of intense study [15, 11, 24, 13, 20], not least because the exact computational complexity of turn-based parity games is still open.

The more complicated variety of graph games is referred to as *concurrent* games. If the vertex of a game is a state in a concurrent game graph, then both players choose simultaneously and independently from a given set of moves, and the pair of chosen moves determines the next state of the game. Such concurrent games—besides being the standard model of game theory for infinitely repeating games of perfect information—arise in computer science applications when the players represent components of a reactive system that interact synchronously, which is often the case [1]. The algorithmics of concurrent games is considerably more difficult than for turn-based games. A principal reason is that while in turn-based games, pure (nonprobabilistic) stationary (memoryless) strategies suffice for winning, in concurrent games, in general players need to use mixed infinite-memory strategies [5, 3]. As a result, few algorithms are available for solving concurrent games. Indeed, for general winning conditions such as parity, and qualitative winning mode, currently the only available algorithm is presented in [5] (a complete version has been submitted to a journal [3]).

Since optimal strategies for concurrent games are in general probabilistic, the classical game-theoretic definition of *qualitative* winning is that the supremum over all player-1 strategies of the infimum over all player-2 strategies of the probability that the resulting path lies in the winning set is 1; in other words, player 1 can win with probability arbitrarily close to 1. In [7, 5], where several modes of qualitative winning have been studied, this is called *limit winning*. A stronger notion of qualitative winning, called *almost-sure* winning, is that player 1 can win with probability 1. (While limit winning and almost-sure winning differ in concurrent games [7], they coincide for turn-based games, where pure strategies suffice for winning.)

In this paper, we provide a *reduction from concurrent parity games to turn-based parity games* for limit-winning. This reduction allows us to transfer all knowledge that has been and will be accumulated for turn-based games, all algorithms and optimizations and tools that are and will become available for turn-based games, to the technically more involved case of concurrent games. For example, as corollaries of our reduction, we obtain new algorithms for solving concurrent parity games that in some cases (e.g., if only a constant number of moves is available for both players at each state) offer better theoretical performance than the currently known algorithm for concurrent parity games.

The only previously known reduction from concurrent to turn-based games is for the special case of almost-sure winning with respect to Büchi and coBüchi conditions [12]. By contrast, our reduction handles the general class of all parity conditions and the general case of limit winning. Our reduction converts every concurrent state $s$ by a turn-based gadget. The gadget is exponential in the number of actions available to each player at $s$, and we establish the equivalence of limit winning in the concurrent game and the turn-based game for parity conditions to obtain the desired result.

## 2 Definitions

In this section we define game structures, strategies, objectives, the limit winning mode and other preliminary definitions.

**Game structures.** We define concurrent game structures and its sub-classes.

*Probability distributions.* For a finite set $A$, a *probability distribution* on $A$ is a function $\delta\colon A \mapsto [0,1]$ such that $\sum_{a\in A}\delta(a) = 1$. We denote the set of probability distributions on $A$ by $\mathcal{D}(A)$. Given $\delta \in \mathcal{D}(A)$, we denote by $\mathrm{Supp}(\delta) = \{x \in A \mid \delta(x) > 0\}$ the *support* of the distribution $\delta$.

*Concurrent game structures.* A concurrent (two-player) *game structure* $\mathcal{G} = \langle S, \mathsf{M}, \Gamma_1, \Gamma_2, \delta\rangle$ consists of the following components.

- A finite state space $S$ and a finite set $\mathsf{M}$ of moves or actions.
- Two move assignments $\Gamma_1, \Gamma_2\colon S \mapsto 2^{\mathsf{M}} \setminus \emptyset$. For $i \in \{1,2\}$, assignment $\Gamma_i$ associates with each state $s \in S$ the nonempty set $\Gamma_i(s) \subseteq \mathsf{M}$ of moves available to player $i$ at state $s$. For technical convenience, we assume that $\Gamma_i(s) \cap \Gamma_j(t) = \emptyset$ unless $i = j$ and $s = t$, for all $i, j \in \{1,2\}$ and $s, t \in S$.
- A probabilistic transition function $\delta\colon S \times \mathsf{M} \times \mathsf{M} \mapsto \mathcal{D}(S)$, which associates with every state $s \in S$ and moves $a_1 \in \Gamma_1(s)$ and $a_2 \in \Gamma_2(s)$ a probability distribution $\delta(s, a_1, a_2) \in \mathcal{D}(S)$ for the successor state.

*Plays.* At every state $s \in S$, player 1 chooses a move $a_1 \in \Gamma_1(s)$, and simultaneously and independently player 2 chooses a move $a_2 \in \Gamma_2(s)$. The game then proceeds to the successor state $t$ with probability $\delta(s, a_1, a_2)(t)$, for all $t \in S$. For all states $s \in S$ and moves $a_1 \in \Gamma_1(s)$ and $a_2 \in \Gamma_2(s)$, we indicate by $\mathit{Dest}(s, a_1, a_2) = \mathrm{Supp}(\delta(s, a_1, a_2))$ the set of possible successors of $s$ when moves $a_1$, $a_2$ are selected. A *path* or a *play* of $\mathcal{G}$ is an infinite sequence $\omega = \langle s_0, s_1, s_2, \ldots\rangle$ of states in $S$ such that for all $k \geq 0$, there are moves $a_1^k \in \Gamma_1(s_k)$ and $a_2^k \in \Gamma_2(s_k)$ such that $s_{k+1} \in \mathit{Dest}(s_k, a_1^k, a_2^k)$. We denote by $\Omega$ the set of all paths.

*Special classes of games.* We distinguish the following subclasses of game structures:

- A game structure $\mathcal{G}$ is *deterministic* if $|\mathit{Dest}(s, a_1, a_2)| = 1$ for all $s \in S$ and all $a_1 \in \Gamma_1(s)$, $a_2 \in \Gamma_2(s)$.
- A game structure $\mathcal{G}$ is *turn-based* if at every state at most one player can choose among multiple moves; that is, for every state $s \in S$ there exists at most one $i \in \{1,2\}$ with $|\Gamma_i(s)| > 1$. Given a turn-based game structure, a state $s$ is a player $i$ state if $|\Gamma_j(s)| = 1$ for $j = \{1,2\} \setminus \{i\}$, i.e., the set of available moves for the other player is singleton.

We define the *size* of the game $\mathcal{G}$ to be equal to the number of entries of the transition function $\delta$; specifically, $|\mathcal{G}| = \sum_{s\in S} \sum_{a\in\Gamma_1(s)} \sum_{b\in\Gamma_2(s)} |\mathit{Dest}(s, a, b)|$.

**Strategies.** A *strategy* for a player is a recipe that describes how to extend a play. Formally, a strategy for player $i \in \{1,2\}$ is a mapping $\pi_i : S^+ \mapsto \mathcal{D}(\mathsf{M})$ that associates with every nonempty finite sequence $x \in S^+$ of states, representing the past history of the game, a probability distribution $\pi_i(x)$ used to select the next move. Thus, the choice of the next move can be history-dependent and randomized. The strategy $\pi_i$ can prescribe only moves that are available to player $i$; that is, for all sequences $x \in S^*$ and states $s \in S$, we require that $\mathrm{Supp}(\pi_i(x \cdot s)) \subseteq \Gamma_i(s)$. We denote by $\Pi_i$ the set of all strategies for player $i \in \{1,2\}$.

Once the starting state $s$ and the strategies $\pi_1$ and $\pi_2$ for the two players have been chosen, the game is reduced to an ordinary stochastic process. Hence, the probabilities of events are uniquely defined, where an *event* $\mathcal{A} \subseteq \Omega$ is a measurable set of paths. For an event $\mathcal{A} \subseteq \Omega$, we denote by $\Pr_s^{\pi_1,\pi_2}(\mathcal{A})$ the probability that a path belongs to $\mathcal{A}$ when the game starts from $s$ and the players use the strategies $\pi_1$ and $\pi_2$.

*Deterministic strategies.* A strategy $\pi$ is *deterministic* if for all $x \in S^+$ there exists $a \in \mathsf{M}$ such that $\pi(x)(a) = 1$. Thus, deterministic strategies are equivalent to functions $S^+ \mapsto \mathsf{M}$. We denote by $\Pi_i^D$ the set of deterministic strategies for player $i$.

**Objectives.** We specify objectives for the players by providing the set of *winning plays* $\Phi \subseteq \Omega$ for each player. In this paper we study only zero-sum games [18, 9], where the objectives of the two players are complementary. In this paper we consider *$\omega$-regular objectives* [21] specified as Rabin-chain (parity) objectives. For a play $\omega = \langle s_0, s_1, s_2, \ldots \rangle \in \Omega$, we define $\mathit{Inf}(\omega) = \{s \in S \mid s_k = s \text{ for infinitely many } k \geq 0\}$ to be the set of states that occur infinitely often in $\omega$. The parity objectives are defined as follows.

- *Rabin-chain (parity) objectives.* For $c, d \in \mathbb{N}$, we let $[c..d] = \{c, c+1, \ldots, d\}$. Let $p : S \mapsto [0..d]$ be a function that assigns a *priority* $p(s)$ to every state $s \in S$, where $d \in \mathbb{N}$. The *Even parity objective* is defined as $\text{Parity}(p) = \{\omega \in \Omega \mid \max\big(p(\mathit{Inf}(\omega))\big) \text{ is even }\}$, and the *Odd parity objective* as $\text{coParity}(p) = \{\omega \in \Omega \mid \max\big(p(\mathit{Inf}(\omega))\big) \text{ is odd }\}$.

**Limit winning mode.** Given an objective $\Phi$, for all initial states $s \in S$, the set of paths $\Phi$ is measurable for all choices of the strategies of the player [23]. Given an initial state $s \in S$ and an objective $\Phi$, we consider the following *winning mode* for player 1: we say that player 1 *wins limit surely* if the player has strategies to win with probability arbitrarily close to 1, or $\sup_{\pi_1 \in \Pi_1} \inf_{\pi_2 \in \Pi_2} \Pr_s^{\pi_1,\pi_2}(\Phi) = 1$. Analogous definitions apply for player 2. We abbreviate the winning mode by *limit*. Using a notation derived from *alternating temporal logic* [1], given a player $i \in \{1, 2\}$, and an objective $\Phi$, we denote by $\langle\langle i \rangle\rangle_{\mathit{limit}}(\Phi)$ the set of states from which player $i$ can win in limit winning mode the game with objective $\Phi$.

We remark that the ability of solving games with Rabin-chain objectives suffices for solving games with respect to arbitrary $\omega$-regular objectives. In fact, we can encode a general $\omega$-regular objective as a deterministic Rabin-chain automaton. By taking the synchronous product of the automaton and the original game, we obtain an (enlarged) game with a Rabin-chain objective [22, 14]. The set of winning states of the original structure can be computed by computing the set of winning states of this enlarged game.

**Mu-calculus, complementation, and levels.** Consider a mu-calculus expression $Y = \mu X . \phi(X)$ over a finite set $S$, where $\phi : 2^S \mapsto 2^S$ is monotonic. The least fixpoint $Y = \mu X . \phi(X)$ of $X = \phi(X)$ is equal to the limit $Y = \lim_{k\to\infty} X_k$, where $X_0 = \emptyset$, and $X_{k+1} = \phi(X_k)$. For every state $s \in Y$, we define the *level* $k \geq 0$ of $s$ in $\mu X . \phi(X)$ to be the integer such that $s \notin X_k$ and $s \in X_{k+1}$. The greatest fixpoint $Y = \nu X . \phi(X)$ of $X = \phi(X)$ is equal to the limit $Y = \lim_{k\to\infty} X_k$, where $X_0 = S$, and $X_{k+1} = \phi(X_k)$. For every state $s \notin Y$, we define the *level* $k \geq 0$ of $s$ in $\nu X . \phi(X)$ to be the integer such that $s \in X_k$ and $s \notin X_{k+1}$. The *height* of a mu-calculus expression $Y = \lambda X . \phi(X)$, where $\lambda \in \{\mu, \nu\}$, is the maximal level of any state in $Y$, i.e., the integer $h$ such that $X_h = \lim_{k\to\infty} X_k$. An expression of height $h$ can be computed in $h + 1$ iterations. Given a mu-calculus expression $Y = \lambda X . \phi(X)$, where $\lambda \in \{\mu, \nu\}$, the complement $\neg Y = S \setminus Y$ of $\lambda$ is given by $\neg Y = \overline{\lambda} X . \neg\phi(\neg X)$, where $\overline{\lambda} = \mu$ if $\lambda = \nu$, and $\overline{\lambda} = \nu$ if $\lambda = \mu$.

**Distributions and one-step transitions.** Given a state $s \in S$, we denote by $\chi_1^s = \mathcal{D}(\Gamma_1(s))$ and $\chi_2^s = \mathcal{D}(\Gamma_2(s))$ the sets of probability distributions over the moves at $s$ available to player 1 and 2, respectively. Moreover, for $s \in S$, $X \subseteq S$, $\xi_1 \in \chi_1^s$, and $\xi_2 \in \chi_2^s$ we denote by

$$P_s^{\xi_1,\xi_2}(X) = \sum_{a\in\Gamma_1(s)} \sum_{b\in\Gamma_2(s)} \sum_{t\in X} \xi_1(a)\xi_2(b)\delta(s,a,b)(t)$$

the one-step probability of a transition into $X$ when players 1 and 2 play at $s$ with distributions $\xi_1$ and $\xi_2$, respectively. Given a state $s$ and distributions $\xi_1 \in \chi_1^s$ and $\xi_2 \in \chi_2^s$ we denote by $Dest(s,\xi_1,\xi_2) = \{t \in S \mid P_2^{\xi_1,\xi_2}(t) > 0\}$ the set of states that have positive probability of transition from $s$ when the players play $\xi_1$ and $\xi_2$ at $s$.

## 3 Limit Winning Set Characterization by $\mu$-Calculus Formula

In this section we present several predecessor operators that are required for the computation of the limit winning set in concurrent games with parity objectives. We then present the $\mu$-calculus formula that uses the predecessor operators to compute the limit winning set in concurrent parity games. Finally, we present the $\mu$-calculus formula that characterizes the predecessor operators. The proofs of the above characterization are available in [3], the results first appeared in [5] and the detailed proofs are available in [3].

**The $\mathrm{Lpre}_1$, $\mathrm{LPreOdd}_1$ and $\mathrm{LPreEven}_1$ operators.** For $s \in S$ and $X, Y \subseteq S$, the $\mathrm{Lpre}_1$ predecessor operator is defined as follows:

$$\mathrm{Lpre}_1(Y,X) = \{s \in S \mid \forall\alpha > 0 \,.\, \exists\xi_1 \in \chi_1^s \,.\, \forall\xi_2 \in \chi_2^s \,.\, \left[P_s^{\xi_1,\xi_2}(X) > \alpha \cdot P_s^{\xi_1,\xi_2}(\neg Y)\right]\} \,.$$

The operator $\mathrm{Lpre}_1(Y,X)$ states that player 1 can choose distributions to ensure that the probability to progress to $X$ can be made arbitrarily large as compared to the probability of escape from $Y$. The $\mathrm{Lpre}_1$ operator is generalized to a $2i$-argument predecessor operator, namely $\mathrm{LPreOdd}_1$ operator, as follows: for $i \geq 0$, and $Y_n, X_n, \ldots, Y_{n-i}, X_{n-i} \subseteq S$, we have

$$\mathrm{LPreOdd}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}) =$$

$$\left\{ s \in S \mid \forall\alpha > 0 \,.\, \exists\xi_1 \in \chi_1^s . \forall\xi_2 \in \chi_2^s . \begin{bmatrix} P_s^{\xi_1,\xi_2}(X_n) > \alpha \cdot P_s^{\xi_1,\xi_2}(\neg Y_n) \\ \bigvee \\ P_s^{\xi_1,\xi_2}(X_{n-1}) > \alpha \cdot P_s^{\xi_1,\xi_2}(\neg Y_{n-1}) \\ \bigvee \\ \vdots \\ \bigvee \\ P_s^{\xi_1,\xi_2}(X_{n-i}) > \alpha \cdot P_s^{\xi_1,\xi_2}(\neg Y_{n-i}) \end{bmatrix} \right\} .$$

The generalization is obtained as follows: the operator $\mathrm{LPreOdd}_1$ requires that for all $\alpha > 0$, there is a player 1 distribution that can ensure the condition similar to $\mathrm{Lpre}_1$ for a disjunction of state pairs $X_{n-j}, Y_{n-j}$ for $0 \leq j \leq i$, and in particular we have $\mathrm{LPreOdd}_1(0, Y_n, X_n) = \mathrm{Lpre}_1(Y_n, X_n)$. The operator $\mathrm{LPreEven}_1$ is defined as follows: for $i \geq 0$ and $Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1} \subseteq S$, we have

$$\text{LPreEven}_1(i, Y_n, X_n, \dots, Y_{n-i}, X_{n-i}, Y_{n-i-1}) =$$

$$\left\{ s \in S \mid \forall \alpha > 0 \,.\, \exists \xi_1 \in \chi_1^s . \forall \xi_2 \in \chi_2^s . \begin{bmatrix} P_s^{\xi_1,\xi_2}(X_n) > \alpha \cdot P_s^{\xi_1,\xi_2}(\neg Y_n) \\ \bigvee \\ P_s^{\xi_1,\xi_2}(X_{n-1}) > \alpha \cdot P_s^{\xi_1,\xi_2}(\neg Y_{n-1}) \\ \bigvee \\ \vdots \\ \bigvee \\ P_s^{\xi_1,\xi_2}(X_{n-i}) > \alpha \cdot P_s^{\xi_1,\xi_2}(\neg Y_{n-i}) \\ \bigvee \\ P_s^{\xi_1,\xi_2}(Y_{n-i-1}) = 1 \end{bmatrix} \right\}.$$

The operator $\text{LPreEven}_1$ requires that there for all $\alpha > 0$, there is player 1 distribution that can ensure that either in the next step all the successor states belong to $Y_{n-i-1}$ or else the conditions of $\text{LPreOdd}_1$ are satisfied. The results of [3] characterizes the limit-winning sets of concurrent parity games by $\mu$-calculus formulas with the predecessor operators defined above. Formally, the following theorem is proved in [3].

**Theorem 1 (Concurrent parity games[3]).** *Given a concurrent game structure $\mathcal{G}$ with a parity objective Parity(p), where $p : S \mapsto \{0, 1, \dots, 2n-1\}$, let $B_i = p^{-1}(i)$ be the set of states with priority $i$. Then $\langle\langle 1 \rangle\rangle_{limit}(\mathit{Parity}(p)) = W$, where $W$ is defined as follows*

$$\nu Y_n.\mu X_n. \cdots \nu Y_1.\mu X_1.\nu Y_0. \begin{bmatrix} B_{2n-1} \cap \mathit{LPreOdd}_1(0, Y_n, X_n) \\ \cup \\ B_{2n-2} \cap \mathit{LPreEven}_1(0, Y_n, X_n, Y_{n-1}) \\ \cup \\ B_{2n-3} \cap \mathit{LPreOdd}_1(1, Y_n, X_n, Y_{n-1}, X_{n-1}) \\ \cup \\ B_{2n-4} \cap \mathit{LPreEven}_1(1, Y_n, X_n, Y_{n-1}, X_{n-1}, Y_{n-2}) \\ \vdots \\ B_1 \cap \mathit{LPreOdd}_1(n-1, Y_n, X_n, \dots, Y_1, X_1) \\ \cup \\ B_0 \cap \mathit{LPreEven}_1(n-1, Y_n, X_n, \dots, Y_1, X_1, Y_0) \end{bmatrix}$$

Our goal is to obtain turn-based reductions to mimic the evaluation of the predecessor operators. In concurrent games edges are labelled by a pair of actions and for every state we will obtain turn-based gadgets containing auxiliary states so that the questions about the predecessor operators on the concurrent games can be answered by evaluating similar questions on the turn-based gadgets. On the concurrent game structures the predecessor operators can be computed as fixed-point expressions over the set of actions and our turn-based gadgets will mimic the fixed-point evaluation. We now present the fixed-point characterization of the predecessor operators.

**The $\mu$-calculus characterization of the predecessor operators.** The definitions of $\text{Lpre}_1$, $\text{LPreOdd}_1$ and $\text{LPreEven}_1$ are not computational. We now present the $\mu$-calculus characterization of the predecessor operators, and the $\mu$-calculus expressions to compute the predecessor operators are over the set $\Gamma_s = \Gamma_1(s) \cup \Gamma_2(s)$. For $X, Y \subseteq S$ and $A \subseteq \Gamma_s$, we define two predicates, $\mathsf{Stay}_i(s, Y, A)$ and $\mathsf{Cover}_i(s, X, A)$ by:

$$\mathsf{Stay}_1(s, Y, A) = \{a \in \Gamma_1(s) \mid \forall b \in \Gamma_2(s) \setminus A \, . \, [Dest(s, a, b) \subseteq Y]\} \quad (1)$$

$$\mathsf{Cover}_1(s, X, A) = \{b \in \Gamma_2(s) \mid \exists a \in \Gamma_1(s) \cap A \, . \, Dest(s, a, b) \cap X \neq \emptyset\} \; ; \quad (2)$$

The set $\mathsf{Stay}_1(s, Y, A) \subseteq \Gamma_1(s)$ consists of the set of player 1 moves $a$ such that for all moves $b$ for player 2 that are not in $A$, the next state given moves $a$ and $b$ is in $Y$ with probability 1. The set $\mathsf{Cover}_1(s, X, A) \subseteq \Gamma_2(s)$ consists of player 2 moves $b$ such that there is a move $a$ for player 1 in $A$ such that the next state given moves $a$ and $b$ is in $X$ with positive probability.

**Lemma 1 ([3])** *For all $X_0 \subseteq Y_0 \subseteq S$, $s \in S$, the following assertion hold: $s \in Lpre_1(Y_0, X_0)$ iff $\Gamma_2(s) \subseteq \mu W \, . \, [\mathsf{Stay}_1(s, Y_0, W) \cup \mathsf{Cover}_1(s, X_0, W)]$.*

**Lemma 2 ([3])** *For all $X_n \subseteq X_{n-1} \subseteq \cdots \subseteq X_{n-i} \subseteq Y_{n-i} \subseteq Y_{n-i+1} \subseteq \cdots \subseteq Y_n$ and $s \in S$, the following assertion hold: let*

$$W_{2i}^* = \mu W_{2i} \, . \, \nu W_{2i-1} \, . \, \cdots \, . \, \mu W_0 \, . \left[ \begin{array}{c} \left(\mathsf{Stay}_1(s, Y_n, W_0) \cap \bigcap_{j=1}^{i} \mathsf{Stay}_1(s, Y_{n-j}, W_{2j-1})\right) \\ \cup \\ \bigcap_{j=0}^{i} \mathsf{Cover}_1(s, X_{n-j}, W_{2j}) \end{array} \right].$$

*We have $s \in LPreOdd_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i})$ iff $\Gamma_2(s) \subseteq W_{2i}^*$.*

**Lemma 3 ([3])** *For all $X_n \subseteq X_{n-1} \subseteq \cdots \subseteq X_{n-i} \subseteq Y_{n-i-1} \subseteq Y_{n-i} \subseteq Y_{n-i+1} \subseteq \cdots \subseteq Y_n$ and $s \in S$, the following assertion hold: let*

$$W_{2i+1}^* = \nu W_{2i+1} \, . \, \mu W_{2i} \, . \, \nu W_{2i-1} \, . \, \cdots \, . \, \mu W_0 \, . \left[ \begin{array}{c} \left(\mathsf{Stay}_1(s, Y_n, W_0) \cap \bigcap_{j=1}^{i} \mathsf{Stay}_1(s, Y_{n-j}, W_{2j-1}) \right. \\ \left. \cap \; \mathsf{Stay}_1(s, Y_{n-i-1}, W_{2i+1})\right) \\ \cup \\ \bigcap_{j=0}^{i} \mathsf{Cover}_1(s, X_{n-j}, W_{2j}) \end{array} \right].$$

*We have $s \in LPreEven_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})$ iff $\Gamma_1(s) \cap W_{2i+1}^* \neq \emptyset$.*

Theorem 1 and Lemmas 1, 2, 3 first appeared in [5], and the detailed proofs are given in [3].

# 4 Reduction for Limit-winning

In this section we present turn-based reduction gadgets for the Lpre, LPreOdd and LPreEven operators and then reduce concurrent game structures to turn-based game structures preserving limit-winning for parity objectives. We first present a basic reduction gadget.

**Basic gadget for reduction.** Given a concurrent game structure $\mathcal{G} = (S, \mathsf{M}, \Gamma_1, \Gamma_2, \delta)$ we construct a basic reduction gadget $\widetilde{\mathrm{gad}}(s, \overline{A}, \overline{B})$ for a state $s \in S$, $\overline{A} \subseteq \Gamma_1(s)$, and $\overline{B} \subseteq \Gamma_2(s)$ as follows:

1. The set of states for the gadget $\widetilde{\text{gad}}(s, \overline{A}, \overline{B})$ is as follows:

$$\begin{array}{rl} & \{(s, A, B, 1) \mid A \subseteq \overline{A}, B \subseteq \overline{B}, B \neq \emptyset\} \\ \cup & \{(s, A_1, A_2, B_1, B_2, 2) \mid A_1, A_2 \subseteq \overline{A}, B_1, B_2 \subseteq \overline{B}, B_1 \neq \emptyset\}. \end{array}$$

   We refer to $(s, \overline{A}, \overline{B}, 1)$ as the starting state of $\widetilde{\text{gad}}(s, \overline{A}, \overline{B})$.

2. The move assignments are as follows: (a) $\widetilde{\Gamma}_1((s, A, B, 1)) = \{(A_i, B_i) \mid A_i \subseteq A, B_i \subseteq B\}$ and $\widetilde{\Gamma}_2((s, A, B, 1)) = \{\bot\}$; i.e., a state $(s, A, B, 1)$ is a player 1 state; and (b) $\widetilde{\Gamma}_1((s, A_1, A_2, B_1, B_2, 2)) = \{\bot\}$ and $\widetilde{\Gamma}_2((s, A_1, A_2, B_1, B_2, 2)) = B_1 \cup \{B_2\}$ if $B_2 \neq \emptyset$ and $B_1$ otherwise, i.e., $(s, A_1, A_2, B_1, B_2, 2)$ is a player 2 state.

   Intuitively at state $(s, A, B, 1)$ player 1 can play a subset $A_i \subseteq A$ of moves and propose to cover $B_i \subseteq B$ of moves. At state $(s, A_1, A_2, B_1, B_2, 2)$ player 2 can check if $B_1$ is indeed covered by $A_1$ and if so it proceeds to the next level by going to state $(s, A_2, B_2, 1)$.

3. We now describe the transition function $\widetilde{\delta}$. For a set $A \subseteq \Gamma_1(s)$ we denote by $\text{unif}(A)$ the uniform distribution over $A$, i.e., $\text{unif}(A)(a) = 0$ for $a \notin A$, and $\frac{1}{|A|}$ otherwise, and for a move $b \in \Gamma_2(s)$, by a slight abuse of notation, we also use $b$ to denote the distribution that deterministically chooses $b$ (i.e., chooses $b$ with probability 1) The transition function is as follows:

$$\begin{array}{rcll} \widetilde{\delta}((s, A, B, 1), (A_i, B_i), \bot) & = & (s, A_i, A \setminus A_i, B_i, B \setminus B_i, 2) & \\ \widetilde{\delta}((s, A_1, A_2, B_1, B_2, 2), \bot, b)(s') & = & P_s^{\text{unif}(\overline{A} \setminus A_2), b}(s') & b \in B_1 \\ \widetilde{\delta}((s, A_1, A_2, B_1, B_2, 2), \bot, \{B_2\}) & = & (s, A_2, B_2, 1) & \end{array}$$

   The states $(s, \emptyset, B, 1)$ are absorbing states that are winning for player 2, i.e., absorbing states assigned odd priority.

   The transition function captures the following idea: at $(s, A, B, 1)$ player 1 can choose $(A_i, B_i)$ to indicate that it covers $B_i$ by playing $A_i$ in the present level and then has $A \setminus A_i$ to cover $B \setminus B_i$ in a top-down fashion in the next level. Intuitively, covering a set $B_i$ of player 2 moves means that player 1 exhibit witness that it can achieve its goal of satisfying the required predecessor operator if player 2 is restricted to choose from the set $B_i$ of moves. Given the game is at state $(s, A, B, 1)$, player 1 has already played the actions $\overline{A} \setminus A$, and effectively it has the set $A_i \cup (\overline{A} \setminus A)$ to cover $B_i$. At $(s, A_1, A_2, B_1, B_2, 2)$ player 2 can either challenge player 1 by playing a move $b \in B_1$ to check if $A_1$ along with actions already played before indeed covers $B_1$, or else can move to the next level with the set of moves for player 1 and player 2 being $A_2$ and $B_2$, respectively. Given player 2 plays a action $b \in B_1$, the transition function is given by the uniform distribution over the actions $\overline{A} \setminus A_2$ (the actions that are already played by player 1) and the action $b$ of player 2. Observe that for $\widetilde{s} = (s, A_1, A_2, B_1, B_2, 2)$ and $b \in B_1$ we have $Dest(\widetilde{s}, \bot, b) = \bigcup_{a \in \overline{A} \setminus A_2} Dest(s, a, b)$. A state $(s, \emptyset, B, 1)$ indicates that player 1 cannot cover all the moves for player 2 and hence it is converted to a loosing state for player 1. Observe that for state $(s, A, B, 1)$ we do not allow $B = \emptyset$: if $B = \emptyset$, then it means player 1 can successfully cover all the moves of player 2 and achieve its goal of satisfying the required predecessor operator.

**Reduction for Lpre.** The gadget $\mathsf{LimOdd}(s)$ for a state $s$ for the Lpre operator is as follows: the state $s$ is converted to a player 1 state with a deterministic transition to the starting state of

$\widetilde{\mathrm{gad}}(s, \Gamma_1(s), \Gamma_2(s))$. Let all the states in $S$ be labelled by a proposition $r$ and all the other states by proposition $\neg r$. We assume that every state $(s, \emptyset, B, 1)$ which are absorbing loosing state for player 1 are labeled by proposition $\ell$. We will follow the same notations for the later reductions as well.

**The $\widetilde{\mathbf{Lpre}}_1$ operator.** We use the *until* operator $\mathcal{U}$ of LTL with the standard semantics. The $\widetilde{\mathrm{Lpre}}_1$ operator is defined as follows: for $X, Y \subseteq S$ we have

$$\widetilde{\mathrm{Lpre}}_1(Y, X) = \left\{ s \in S \mid \exists \alpha \in \mathbb{R}_{>0}.\exists \widetilde{\pi}_1.\forall \widetilde{\pi}_2. \begin{bmatrix} \mathrm{Pr}_s^{\widetilde{\pi}_1, \widetilde{\pi}_2}\big( \ (\neg r\, \mathcal{U}\, Y)\big) = 1 \\ \wedge \\ \mathrm{Pr}_s^{\widetilde{\pi}_1, \widetilde{\pi}_2}\big( \ (\neg r\, \mathcal{U}\, X)\big) \geq \alpha \end{bmatrix} \text{ in } \mathsf{LimOdd}(s). \right\}$$

**Lemma 4** *For all* $X_0 \subseteq Y_0 \subseteq S$ *we have* $Lpre_1(Y_0, X_0) = \widetilde{Lpre}_1(Y_0, X_0)$.

**Proof.** To prove the desired result we consider the fixed-point characterization of Lpre by Lemma 1. We prove inclusion in both directions.

1. We first show that $\mathrm{Lpre}_1(Y_0, X_0) \subseteq \widetilde{\mathrm{Lpre}}_1(Y_0, X_0)$. If $s \in \mathrm{Lpre}_1(Y_0, X_0)$, then by Lemma 1 we have $\Gamma_2(s) \subseteq \mu W \,.\, \big[\mathsf{Stay}_1(s, Y_0, W) \cup \mathsf{Cover}_1(s, X_0, W)\big]$. Let

$$W_{-1} = \emptyset; \qquad \text{for } i \geq 0, \quad W_i = \mathsf{Stay}_1(s, Y_0, W_{i-1}) \cup \mathsf{Cover}_1(s, X_0, W_{i-1}).$$

   Then we have $\mu W \,.\, \big[\mathsf{Stay}_1(s, Y_0, W) \cup \mathsf{Cover}_1(s, X_0, W)\big] = \bigcup_{i \geq 0} W_i$; and hence $\Gamma_2(s) \subseteq \bigcup_{i \geq 0} W_i$. For $i \geq 0$, let $A_i = (W_i \setminus W_{i-1}) \cap \Gamma_1(s)$ and $B_i = (W_i \setminus W_{i-1}) \cap \Gamma_2(s)$. We have the following properties of the moves. Consider a move $b$ in $B_i$ (i.e., a player 2 move in $W_i \setminus W_{i-1}$).

   (a) *Property 1.* Since $b \in W_i$, by definition of $\mathsf{Cover}_1(s, X_0, W_{i-1})$ there exists $a \in W_{i-1}$ such that $Dest(s, a, b) \cap X_0 \neq \emptyset$.

   (b) *Property 2.* By definition of $\mathsf{Stay}_1(s, Y_0, W_{i-1})$ for all $a \in W_i$, for all $b' \in \Gamma_2(s) \setminus W_{i-1}$ we have $Dest(s, a, b') \subseteq Y_0$. Since $b \in W_i \setminus W_{i-1}$ (i.e., $b \notin W_{i-1}$), it follows that for all $a \in W_i$ we have $Dest(s, a, b) \subseteq Y_0$.

   The strategy $\widetilde{\pi}_1$ for player 1 as a witness that $s \in \widetilde{\mathrm{Lpre}}_1(Y_0, X_0)$ is obtained as follows:

   - $\widetilde{\pi}_1((s, \Gamma_1(s), \Gamma_2(s), 1)) = (A_0, B_0)$; and
   - for $i > 0$, $\widetilde{\pi}_1((s, \Gamma_1(s) \setminus W_{i-1}, \Gamma_2(s) \setminus W_{i-1}, 1)) = (A_i, B_i)$.

   For $i \geq 0$, for the state $\widetilde{s} = (s, A_i, \Gamma_1(s) \setminus W_i, B_i, \Gamma_2(s) \setminus W_i, 2)$, the following assertions hold: (a) for all $b \in B_i$ we have $Dest(\widetilde{s}, \perp, b) \cap X_0 = (\bigcup_{a \in W_i} Dest(s, a, b)) \cap X_0 \neq \emptyset$ (by property 1); (b) for all $b \in B_i$ we have $Dest(\widetilde{s}, \perp, b) = \bigcup_{a \in W_i} Dest(s, a, b) \subseteq Y_0$ (by property 2). Since $\Gamma_2(s) \subseteq \bigcup_{i \geq 0} W_i$ it follows that $\widetilde{\pi}_1$ ensures that absorbing states $(s, \emptyset, B, 1)$ are never reached. The assertions that follow from property 1 and property 2 ensure that the set $Y_0$ is reached with probability 1 and the set $X_0$ is reached with positive probability. The desired result follows.

2. We now show that $\widetilde{\mathrm{Lpre}}_1(Y_0, X_0) \subseteq \mathrm{Lpre}_1(Y_0, X_0)$. If there is witness strategy $\widetilde{\pi}_1$ to satisfy $\widetilde{\mathrm{Lpre}}_1(Y_0, X_0)$, then since the reduction gadget is turn-based there is also a deterministic witness strategy $\widetilde{\pi}_1^D$. We inductively define the following pairs of actions: (a) let $\widetilde{\pi}_1^D((s, \Gamma_1(s), \Gamma_2(s), 1)) = (\widetilde{A}_0, \widetilde{B}_0)$; and (b) for $k > 0$, let $\widetilde{\pi}_1^D((s, \Gamma_1(s) \setminus \bigcup_{i<k} \widetilde{A}_i, \Gamma_2(s) \setminus \bigcup_{i<k} \widetilde{B}_i, 1)) = (\widetilde{A}_k, \widetilde{B}_k)$. Then for a move $b \in \widetilde{B}_k$ the following assertions hold:

(a) *Condition 1.* exists $a \in \bigcup_{i<k} \widetilde{A}_i$ such that $Dest(s,a,b) \cap X_0 \neq \emptyset$; and

(b) *Condition 2.* for all $a \in \bigcup_{i<k} \widetilde{A}_i$ we have $Dest(s,a,b) \subseteq Y_0$.

The above claim is proved as follows. Assume towards contradiction, that there exists $b^* \in \widetilde{B}_k$ such that one of the above two conditions do not hold. Given the strategy $\widetilde{\pi}_1^D$ player 2 can ensure that the state $\widetilde{s} = (s, \widetilde{A}_k, \Gamma_1(s) \setminus \bigcup_{i \le k} \widetilde{A}_i, \widetilde{B}_k, \Gamma_2(s) \setminus \bigcup_{i \le k} \widetilde{B}_i, 2)$ is reached with probability 1 (by choosing for a state $(s, \widetilde{A}_j, \Gamma_1(s) \setminus \bigcup_{i \le j} \widetilde{A}_i, \widetilde{B}_j, \Gamma_2(s) \setminus \bigcup_{i \le j} \widetilde{B}_i, 2)$ the successor $(s, \Gamma_1(s) \setminus \bigcup_{i \le j} \widetilde{A}_i, \Gamma_2(s) \setminus \bigcup_{i \le j} \widetilde{B}_i, 1)$, for $j < k$); and then play $b^*$ to ensure that either (a) $X_0$ is not reached with positive probability (if Condition 1 fails for $b^*$) or (b) $Y_0$ is not reached with probability 1 (if Condition 2 fails for $b^*$). This would contradict that $\widetilde{\pi}_1^D$ is a witness that $s \in \widetilde{\mathrm{Lpre}}_1(Y_0, X_0)$. Hence it follows that Condition 1 and Condition 2 holds for all $b \in \widetilde{B}_k$. The strategy $\widetilde{\pi}_1^D$ must also ensure that the absorbing states $(s, \emptyset, B, 1)$ are never reached and hence $\Gamma_2(s) \subseteq \bigcup_{k \ge 0} \widetilde{B}_k$. We now produce witness distributions to show that $s \in \mathrm{Lpre}_1(Y_0, X_0)$. Given $\varepsilon > 0$, consider the distribution $\xi_1[\varepsilon]$ in the concurrent game $\mathcal{G}$ that plays moves in $\widetilde{A}_k$ with probability proportional to $\varepsilon^k$. Consider a move $b \in \widetilde{B}_k$. Given player 2 plays move $b$ we have:

- the probability of going to $X_0$ is proportional to at least $\varepsilon^{k-1}$, since for some move $a \in \bigcup_{i<k} \widetilde{A}_i$ we have $Dest(s,a,b) \cap X_0 \neq \emptyset$ (by Condition 1); and
- the probability of leaving $Y_0$ is at most proportional to $\varepsilon^k$, since for all moves $a \in \bigcup_{i<k} \widetilde{A}_i$ we have $Dest(s,a,b) \subseteq Y_0$ (by Condition 2).

It follows that for all distributions $\xi_2 \in \chi_2^s$, the ratio of the probability of going to $X_0$ as compared to the probability of leaving $Y_0$ is proportional to at least $\frac{1}{\varepsilon}$. Since $\varepsilon > 0$ is arbitrary, it follows that $s \in \mathrm{Lpre}_1(Y_0, X_0)$. ∎

**The $\widetilde{\mathbf{LPreOdd}}_1$ operator.** Similar to the definition the $\widetilde{\mathrm{Lpre}}_1$ operator we use the *until* operator $\mathcal{U}$ of LTL to define the $\widetilde{\mathrm{LPreOdd}}_1$ operator. The $\widetilde{\mathrm{LPreOdd}}_1$ operator is defined as follows: for

$Y_n, X_n, \ldots, Y_{n-i}, X_{n-i} \subseteq S$ we have

$$\widetilde{\mathrm{LPreOdd}}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}) =$$

$$\left\{ s \in S \mid \exists \alpha \in \mathbb{R}_{>0}. \exists \widetilde{\pi}_1. \forall \widetilde{\pi}_2. \left[ \begin{array}{c} \left( \mathrm{Pr}_s^{\widetilde{\pi}_1,\widetilde{\pi}_2}( \ (\neg r\, \mathcal{U}\, Y_n)) = 1 \right. \\ \wedge \\ \left. \mathrm{Pr}_s^{\widetilde{\pi}_1,\widetilde{\pi}_2}( \ (\neg r\, \mathcal{U}\, X_n)) \geq \alpha \right) \\ \bigvee \\ \left( \mathrm{Pr}_s^{\widetilde{\pi}_1,\widetilde{\pi}_2}( \ (\neg r\, \mathcal{U}\, Y_{n-1})) = 1 \right. \\ \wedge \\ \left. \mathrm{Pr}_s^{\widetilde{\pi}_1,\widetilde{\pi}_2}( \ (\neg r\, \mathcal{U}\, X_{n-1})) \geq \alpha \right) \\ \bigvee \\ \vdots \\ \bigvee \\ \left( \mathrm{Pr}_s^{\widetilde{\pi}_1,\widetilde{\pi}_2}( \ (\neg r\, \mathcal{U}\, Y_{n-i})) = 1 \right. \\ \wedge \\ \left. \mathrm{Pr}_s^{\widetilde{\pi}_1,\widetilde{\pi}_2}( \ (\neg r\, \mathcal{U}\, X_{n-i})) \geq \alpha \right) \end{array} \right] \text{ in } \mathsf{LimOdd}(s). \right\}$$

We extend the results of Lemma 4 to $\mathrm{LPreOdd}_1$ operators, and for simplicity we present the details of the result for $\mathrm{LPreOdd}_1(1, Y_1, X_1, Y_0, X_0)$.

**Lemma 5** *For all $X_1 \subseteq X_0 \subseteq Y_0 \subseteq Y_1 \subseteq S$, we have $LPreOdd_1(1, Y_1, X_1, Y_0, X_0) = \widetilde{LPreOdd}_1(1, Y_1, X_1, Y_0, X_0)$.*

**Proof.** We prove inclusion in both directions.

1. We show $\mathrm{LPreOdd}_1(1, Y_1, X_1, Y_0, X_0) \subseteq \widetilde{\mathrm{LPreOdd}}_1(1, Y_1, X_1, Y_0, X_0)$. We will use the $\mu$-calculus characterization of Lemma 2. If $s \in \mathrm{LPreOdd}_1(1, Y_1, X_1, Y_0, X_0)$, then let

$$W_2^* = \mu W_2 \,.\, \nu W_1 \,.\, \mu W_0 \,.\, \left[ \begin{array}{c} \mathsf{Stay}_1(s, Y_1, W_0) \cap \mathsf{Stay}_1(s, Y_0, W_1) \\ \cup \\ \mathsf{Cover}_1(s, X_1, W_0) \cap \mathsf{Cover}_1(s, X_0, W_2) \end{array} \right].$$

We have $\Gamma_2(s) \subseteq W_2^*$. We first analyze the computation of $W_2^*$. The set $W_2^*$ of moves is obtained as follows:

$$\emptyset = W_2^0 \subseteq W_2^1 \subseteq W_2^2 \subseteq \cdots \subseteq W_2^{\ell-1} \subseteq W_2^{\ell} = W_2^{\ell+1} = W_2^*;$$

and the set $W_2^{i+1}$ is obtained from $W_2^i$ as follows:

$$W_2^{i+1} = \nu W_1 \,.\, \mu W_0 \,.\, \left[ \begin{array}{c} \mathsf{Stay}_1(s, Y_1, W_0) \cap \mathsf{Stay}_1(s, Y_0, W_1) \\ \cup \\ \mathsf{Cover}_1(s, X_1, W_0) \cap \mathsf{Cover}_1(s, X_0, W_2^i) \end{array} \right].$$

Alternatively we have

$$W_2^{i+1} = \mu W_0 \, . \left[ \begin{array}{c} \mathsf{Stay}_1(s, Y_1, W_0) \cap \mathsf{Stay}_1(s, Y_0, W_2^{i+1}) \\ \cup \\ \mathsf{Cover}_1(s, X_1, W_0) \cap \mathsf{Cover}_1(s, X_0, W_2^{i}) \end{array} \right].$$

Equivalently, we can characterize the computation of $W_2^{i+1}$ as follows:

$$\begin{array}{rcl} W_2^{i+1,0} & = & W_2^i \;\cup\; \mathsf{Cover}_1(s, X_0, W_2^i); \\ W_2^{i+1,j+1} & = & W_2^{i+1,j} \;\cup\; (\mathsf{Stay}_1(s, Y_1, W_2^{i+1,j}) \cap \mathsf{Stay}_1(s, Y_0, W_2^{i+1})) \cup \mathsf{Cover}_1(s, X_1, W_2^{i+1,j}). \end{array}$$

**Properties.** We now describe the following key properties of the moves.

(a) *Property 1.* For all $b \in (W_2^{i+1,0} \setminus W_2^i) \cap \Gamma_2(s)$, there exists $a \in W_2^i \cap \Gamma_1(s)$ such that $Dest(s, a, b) \cap X_0 \neq \emptyset$ (by property of $\mathsf{Cover}_1(s, X_0, W_2^i)$).

(b) *Property 2.* For all $b \in (W_2^{i+1,j+1} \setminus W_2^{i+1,j}) \cap \Gamma_2(s)$, there exists $a \in W_2^{i+1,j} \cap \Gamma_1(s)$ such that $Dest(s, a, b) \cap X_1 \neq \emptyset$ (by property of $\mathsf{Cover}_1(s, X_1, W_2^{i+1,j})$).

(c) *Property 3.* For all $a \in W_2^{i+1} \cap \Gamma_1(s)$, for all $b \in \Gamma_2(s) \setminus W_2^{i+1}$ we have $Dest(s, a, b) \subseteq Y_0 \subseteq Y_1$ (by property of $\mathsf{Stay}_1(s, Y_0, W_2^{i+1})$).

(d) *Property 4.* For all $a \in W_2^{i+1,j+1} \cap \Gamma_1(s)$, for all $b \in \Gamma_2(s) \setminus W_2^{i+1,j}$ we have $Dest(s, a, b) \subseteq Y_1$ (by property of $\mathsf{Stay}_1(s, Y_1, W_2^{i+1,j})$).

We now define the following move sets for player 1 and player 2 at $s$; (a) for $i \geq 0$, let

$$A^{i+1,0} = (W_2^{i+1,0} \setminus W_2^i) \cap \Gamma_1(s) \quad B^{i+1,0} = (W_2^{i+1,0} \setminus W_2^i) \cap \Gamma_2(s);$$

and (b) for $i \geq 0$ and $j \geq 1$, let

$$A^{i+1,j} = (W_2^{i+1,j} \setminus W_2^{i+1,j-1}) \cap \Gamma_1(s) \quad B^{i+1,j} = (W_2^{i+1,j} \setminus W_2^{i+1,j-1}) \cap \Gamma_2(s).$$

We now construct a strategy $\widetilde{\pi}_1$ for player 1 to witness that $s \in \widetilde{\mathrm{LPreOdd}}_1(1, Y_1, X_1, Y_0, X_0)$. The strategy $\widetilde{\pi}_1$ is as follows:

(a) $\widetilde{\pi}_1\big((s, \Gamma_1(s), \Gamma_2(s), 1)\big) = (A^{1,0}, B^{1,0})$;

(b) for $i \geq 1$, let $\widetilde{\pi}_1\big((s, \Gamma_1(s) \setminus W_2^i, \Gamma_2(s) \setminus W_2^i, 1)\big) = (A^{i+1,0}, B^{i+1,0})$; and

(c) for $i \geq 1$ and $j \geq 0$, let $\widetilde{\pi}_1\big((s, \Gamma_1(s) \setminus W_2^{i,j}, \Gamma_2(s) \setminus W_2^{i,j}, 1)\big) = (A^{i,j}, B^{i,j})$, where $W_2^{i,j}$ is a strict subset of $W_2^{i+1}$.

Given the strategy we have the following case analysis.

(a) *Case 1.* For a state $\widetilde{s} = (s, A^{i+1,0}, \Gamma_1(s) \setminus W_2^{i+1,0}, B^{i+1,0}, \Gamma_2(s) \setminus W_2^{i+1,0}, 2)$, for all $b \in B^{i+1,0}$ we have (a) $Dest(\widetilde{s}, \perp, b) \cap X_0 \neq \emptyset$ (by Property 1); and (b) $Dest(\widetilde{s}, \perp, b) \subseteq Y_0 \subseteq Y_1$ (by Property 3).

(b) *Case 2.* For a state $\widetilde{s} = (s, A^{i,j}, \Gamma_2(s) \setminus W_2^{i,j}, B^{i,j}, \Gamma_1(s) \setminus W_2^{i,j}, 2)$, for all $b \in B^{i,j}$ we have (a) $Dest(\widetilde{s}, \perp, b) \cap X_1 \neq \emptyset$ (by Property 2); and (b) $Dest(\widetilde{s}, \perp, b) \subseteq Y_1$ (by Property 4).

Given the strategy $\widetilde{\pi}_1$ consider a counter-strategy for player 2, and since the reduction is a turn-based gadget it suffices to consider deterministic counter-strategy for player 2. From the above case analysis the following assertions hold: (a) for a player-2 state $\widetilde{s}$ in the gadget, if player 2 plays any action $b \in \Gamma_2(s)$, then the set $Y_1$ is reached with probability 1; (b) if on reaching a player-2 state $\widetilde{s}$ in the gadget, player 2 plays an action $b \in \Gamma_2(s)$, satisfying case 2 above, then $X_1$ is reached with positive probability; and (c) if on reaching a player-2 state $\widetilde{s}$ in the gadget, player 2 plays an action $b \in \Gamma_2(s)$ satisfying case 1 above, then the set $Y_0$ is reached with probability 1 and the set $X_0$ is reached with positive probability. Since $\Gamma_2(s) \subseteq W_2^*$, it follows that $\widetilde{\pi}_1$ is a witness that $s \in \widetilde{\text{LPreOdd}}_1(1, Y_1, X_1, Y_0, X_0)$.

2. We now show the inclusion in other direction. If there is witness strategy $\widetilde{\pi}_1$ to satisfy $\widetilde{\text{LPreOdd}}_1(1, Y_1, X_1, Y_0, X_0)$, then since the reduction gadget is turn-based there is also a deterministic witness strategy $\widetilde{\pi}_1^D$. We inductively define the following pairs of actions: (a) let $\widetilde{\pi}_1^D((s, \Gamma_1(s), \Gamma_2(s), 1) = (\widetilde{A}_0, \widetilde{B}_0)$; and (b) for $k > 0$, let $\widetilde{\pi}_1^D((s, \Gamma_1(s) \setminus \bigcup_{i<k} \widetilde{A}_i, \Gamma_2(s) \setminus \bigcup_{i<k} \widetilde{B}_i, 1)) = (\widetilde{A}_k, \widetilde{B}_k)$. Then for a move $b \in \widetilde{B}_k$ the following assertions hold:

   (a) *Condition 1.* either (i) exists $a \in \bigcup_{i<k} \widetilde{A}_i$ such that $Dest(s, a, b) \cap X_1 \neq \emptyset$ or (ii) exists $a \in \bigcup_{i<k} \widetilde{A}_i$ such that $Dest(s, a, b) \cap X_0 \neq \emptyset$ and for all $a \in \bigcup_{i<k} \widetilde{A}_i$ we have $Dest(s, a, b) \subseteq Y_0 \subseteq Y_1$; and

   (b) *Condition 2.* for all $a \in \bigcup_{i<k} \widetilde{A}_i$ we have $Dest(s, a, b) \subseteq Y_1$.

   If one of the above claim fails, then similar to Lemma 5, it can be contradicted that $\widetilde{\pi}_1^D$ is a witness strategy. The strategy $\widetilde{\pi}_1^D$ must also ensure that the absorbing states $(s, \emptyset, B, 1)$ are never reached and hence $\Gamma_2(s) \subseteq \bigcup_{k \geq 0} \widetilde{B}_k$. We now produce witness distributions to show that $s \in \text{LPreOdd}_1(1, Y_1, X_1, Y_0, X_0)$. Given $\varepsilon > 0$, consider the distribution $\xi_1[\varepsilon]$ in the concurrent game $\mathcal{G}$ that plays moves in $\widetilde{A}_k$ with probability proportional to $\varepsilon^k$. Consider a distribution $\xi_2 \in \chi_2^s$ for player 2. Suppose for all $b \in \text{Supp}(\xi_2)$ condition 1.(ii) is satisfied, then it follows that for all $b \in \text{Supp}(\xi_2)$, there exists $j$, such that for some $a \in \bigcup_{i<j} \widetilde{A}_i$ we have $Dest(s, a, b) \cap X_0 \neq \emptyset$ and for all $a \in \bigcup_{i<j} \widetilde{A}_i$ we have $Dest(s, a, b) \subseteq Y_0$. Hence the ratio of the probability of going to $X_0$ as compared to leaving $Y_0$ is proportional to at least $\frac{1}{\varepsilon}$. Otherwise, there exists $b^* \in \text{Supp}(\xi_2)$ such that condition 1.(i) holds. In this case, for all $b \in \widetilde{B}_k$, for all $a \in \bigcup_{i<k} \widetilde{A}_i$ we have $Dest(s, a, b) \subseteq Y_1$ and for all $b \in \widetilde{B}_k$ that satisfies condition 1.(i), for some $a \in \bigcup_{i<k} \widetilde{A}_i$ we have $Dest(s, a, b) \cap X_1 \neq \emptyset$. Hence the ratio of the probability of going to $X_1$ as compared to leaving $Y_1$ is proportional to at least $\frac{1}{\varepsilon}$. Since $\varepsilon > 0$ is arbitrary, it follows that $s \in \text{LPreOdd}_1(1, Y_1, X_1, Y_0, X_0)$. ∎

The result of Lemma 5 can be generalized to give us the following result: the argument is similar to Lemma 5 and in the general case the argument is extended using the ranking function of general $\mu$-calculus formula.

**Lemma 6** *For all $X_n \subseteq X_{n-1} \subseteq \cdots \subseteq X_{n-i} \subseteq Y_{n-i} \subseteq Y_{n-i+1} \subseteq \cdots \subseteq Y_n$ we have $LPreOdd_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}) = \widetilde{LPreOdd}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i})$.*

**Reduction for $\text{LPreEven}_1$.** The reduction of a state $s$ to turn-based gadget $\mathsf{LimEven}(s)$ for the $\text{LPreEven}_1$ operator requires one more level as compared to the $\text{LPreOdd}_1$ operator. The idea is as follows.

- *Set of states.* In the gadget $\mathsf{LimEven}(s)$ the starting state is $s$ and $s$ is player 1 state. The gadget $\mathsf{LimEven}(s)$ along with $s$ has the gadgets $\widetilde{\text{gad}}(s, A, B)$ for $A \subseteq \Gamma_1(s)$ and $B \subseteq \Gamma_2(s)$; and there are also states of the form $(s, A, B)$ where $A \subseteq \Gamma_1(s)$ and $B \subseteq \Gamma_2(s)$ and each state $(s, A, B)$ is a player 2 state.

- *Move assignment.* We have $\widetilde{\Gamma}_1(s) = \{(A, B) \mid A \subseteq \Gamma_1(s), B \subseteq \Gamma_2(s), A \neq \emptyset\}$ and $\widetilde{\Gamma}_2(s) = \{\perp\}$; and $\widetilde{\Gamma}_1((s, A, B)) = \{\perp\}$ and $\widetilde{\Gamma}_2((s, A, B)) = (\Gamma_2(s) \setminus B) \cup \{B\}$.

- *Transition function.* We have $\widetilde{\delta}(s, (A, B), \perp) = (s, A, B)$; $\widetilde{\delta}((s, A, B), \perp, b)(s') = P_s^{\text{unif}(A), b}(\{s'\})$ for $b \in \widetilde{\Gamma}_2((s, A, B))$ and $\widetilde{\delta}((s, A, B), \perp, B) = (s, A, B, 1)$, i.e., goes to the starting state of $\widetilde{\text{gad}}(s, A, B)$.

**The $\widetilde{\textbf{LPreEven}}_1$ operator.** The $\widetilde{\text{LPreEven}}_1$ operator is defined in a fashion similar to the $\widetilde{\text{LPreOdd}}_1$ operator: for $Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1} \subseteq S$ we have

$$\widetilde{\text{LPreEven}}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1}) =$$

$$\Bigg\{ s \in S \mid \exists (A, B).\ A \subseteq \Gamma_1(s), B \subseteq \Gamma_2(s), \forall b \in \Gamma_2(s) \setminus B, \text{(i) either } \bigcup_{a \in A} Dest(s, a, b) \subseteq Y_{n-i-1},$$

$$\text{or for some } 0 \leq j_b \leq i.\ \bigcup_{a \in A} Dest(s, a, b) \subseteq Y_{n-i+j_b} \text{ and } \bigcup_{a \in A} Dest(s, a, b) \cap X_{n-i+j_b} \neq \emptyset;$$

$$\text{and (ii) the starting state of } \widetilde{\text{gad}}(s, A, B) \text{ satisfies } \widetilde{\text{LPreOdd}}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}) \Bigg\}$$

**Lemma 7** *For all $X_n \subseteq X_{n-1} \subseteq \cdots \subseteq X_{n-i} \subseteq Y_{n-i-1} \subseteq Y_{n-i} \subseteq Y_{n-i+1} \subseteq \cdots \subseteq Y_n$ we have $LPreEven_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1}) = \widetilde{LPreEven}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})$.*

**Proof.** We prove inclusion in both directions.

1. We first show that $\text{LPreEven}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1}) \subseteq \widetilde{\text{LPreEven}}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})$. If $s \in \text{LPreEven}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})$, then from the proofs of the $\mu$-calculus characterization of $\text{LPreOdd}_1$ and $\text{LPreEven}_1$ the following property follows (see [3]): there exists $(A^*, B^*)$ with $A^* \subseteq \Gamma_1(s)$ and $B^* \subseteq \Gamma_2(s)$ such that (a) for all $b \in \Gamma_2(s) \setminus B^*$ and for all $a \in A^*$ we have $Dest(s, a, b) \subseteq Y_{n-i-1}$; and (b) if player 2's move are restricted to $B^*$, then player 1 can ensure $\text{LPreOdd}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i})$. The witness $(A^*, B^*)$ is obtained as follows. Let $W^*_{2i+1}$ be the result of the $\mu$-calculus formula over moves for Lemma 3, then $A^* = W^*_{2i+1} \cap \Gamma_1(s)$ and $B^* = W^*_{2i+1} \cap \Gamma_2(s)$. The witness strategy for player 1 to show $s \in \widetilde{\text{LPreEven}}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})$ is as follows: the strategy chooses $(A^*, B^*)$ at $s$, and then plays a strategy $\widetilde{\pi}_1$ from the starting state of $\widetilde{\text{gad}}(s, A^*, B^*)$ to satisfy $\widetilde{\text{LPreOdd}}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i})$ (the existence of such a witness strategy $\widetilde{\pi}_1$ follows from Lemma 6 and the fact that given player 2's move are restricted to $B^*$, then player 1 can ensure $\text{LPreOdd}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i})$). It follows that $s \in \widetilde{\text{LPreEven}}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})$.

2. We now prove the other inclusion to complete the proof. If there is witness strategy $\widetilde{\pi}_1$ to satisfy $\widetilde{\text{LPreEven}}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})$, then since the reduction gadget is turn-based there is also a deterministic witness strategy $\widetilde{\pi}_1^D$. We inductively define the following pairs of actions: (a) let $\widetilde{\pi}_1^D(s) = (\widetilde{A}, \widetilde{B})$; and (b) let $\widetilde{\pi}_1^D((s, \widetilde{A}, \widetilde{B}, 1)) = (\widetilde{A}_0, \widetilde{B}_0)$; and (c) for $k > 0$, let $\widetilde{\pi}_1^D((s, \widetilde{A} \setminus \bigcup_{i<k} \widetilde{A}_i, \widetilde{B} \setminus \bigcup_{i<k} \widetilde{B}_i, 1)) = (\widetilde{A}_k, \widetilde{B}_k)$. Then the following assertions hold.

   (a) *Condition 1.* For all moves $b \in \Gamma_2(s) \setminus \widetilde{B}$, either (i) $\bigcup_{a \in A} Dest(s, a, b) \subseteq Y_{n-i-1} \subseteq Y_{n-i} \subseteq \ldots \subseteq Y_n$, or (ii) for some $0 \leq j_b \leq i$ we have $\bigcup_{a \in A} Dest(s, a, b) \subseteq Y_{n-i+j_b}$ and $\bigcup_{a \in A} Dest(s, a, b) \cap X_{n-i+j_b} \neq \emptyset$.

   (b) *Condition 2.* For all $b \in \widetilde{B}_k$, there exists $0 \leq j_b \leq i$, such that (i) for all $a \in \bigcup_{i<k} \widetilde{A}_i$ we have $Dest(s, a, b) \subseteq Y_{n-i+j_b}$, and (ii) for some $a \in \bigcup_{i<k} \widetilde{A}_i$ we have $Dest(s, a, b) \cap X_{n-i+j_b} \neq \emptyset$.

   If one of the above claim fails, then it can be contradicted that $\widetilde{\pi}_1^D$ is a witness strategy. The strategy $\widetilde{\pi}_1^D$ must also ensure that the absorbing states $(s, \emptyset, B, 1)$ are never reached and hence $\Gamma_2(s) \subseteq \bigcup_{k \geq 0} \widetilde{B}_k$. Let $\widehat{B}$ be the moves of player 2 that satisfies Condition 1.(i), and $\overline{B} = \Gamma_2(s) \setminus \widehat{B}$. Observe that for a move $b \in \overline{B}$ if Condition 1.(ii) or Condition 2 is satisfied for $1 \leq j_b \leq i$, then it also holds for $j$ such that $j_b \leq j \leq i$. Let $j^* \geq 0$ be such that for all moves $b \in \overline{B}$ Condition 1.(ii) or Condition 2 are satisfied for $j^*$ (simply choose $j^*$ to be the maximum of $j_b$ for $b \in \overline{B}$). The witness distributions to show that $s \in \text{LPreEven}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})$ is as follows: given $\varepsilon > 0$, consider the distribution $\xi_1[\varepsilon]$ in the concurrent game $\mathcal{G}$ such that $\text{Supp}(\xi_1[\varepsilon]) = \widetilde{A}$ and that plays moves in $\widetilde{A}_k$ with probability proportional to $\varepsilon^k$. Consider a counter-distribution $\xi_2$ for player 2: the following case analysis completes the proof.

   - If $\text{Supp}(\xi_2) \subseteq \widehat{B}$, then from $s$ the successor states given $\xi_1[\varepsilon]$ and $\xi_2$ are in $Y_{n-i-1}$;
   - otherwise, for $j^*$ the ratio of the probability of going to $X_{n-j^*}$ as compared to leaving $Y_{n-j^*}$ is proportional to at least $\frac{1}{\varepsilon}$.

   Since $\varepsilon > 0$ is arbitrary, it follows that $s \in \text{LPreEven}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-1})$. The result follows. ∎

**Reduction for limit-winning.** Given a concurrent game structure $\mathcal{G}$ with a priority function $p$, we construct a turn-based game structure $\widetilde{\mathcal{G}}_L$ with a priority function $\widetilde{p}$ as follows.

- *Reduction for odd states.* For a state $s$ such that $p(s)$ is odd, $s$ is replaced by the gadget $\mathsf{LimOdd}(s)$, the priority $\widetilde{p}$ is assigned as follows: $\widetilde{p}(s) = p(s)$ and for a state $\widetilde{s}$ we have $\widetilde{p}(\widetilde{s}) = p(s)$. Observe that it is taken care that absorbing states of the form $(s, \emptyset, B, 1)$ are assigned an odd priority.

- *Reduction for even states.* For a state $s$ such that $p(s)$ is even, $s$ is replaced by the gadget $\mathsf{LimEven}(s)$, the priority $\widetilde{p}$ is assigned as follows: (i) $\widetilde{p}(s) = p(s)$, (ii) for $\widetilde{s} = (s, A, B)$ we have $\widetilde{p}(\widetilde{s}) = p(s)$, and (iii) for all other states $\widetilde{s}$ in the gadget we have $\widetilde{p}(\widetilde{s}) = p(s) + 1$. We briefly explain the reason of assigning priority $p(s) + 1$ to certain states in the gadget. The proof of Lemma 7 shows that $s \in \widetilde{\text{LPreEven}}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i+1})$ can be characterized as follows: from $s$ player 1 can play a pair of action sets such that in the immediate successor either player 2 plays in a way such that the condition for $\widetilde{\text{LPreEven}}_1$ holds, or else from the next states player 1 satisfies $\widetilde{\text{LPreOdd}}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i})$. The correspondence of $\widetilde{\text{LPreOdd}}_1$ and $\text{LPreOdd}_1$ operator, and the fact that in the $\mu$-calculus characterization of winning set (by the $\mu$-calculus formula of Theorem 1) if $\text{LPreEven}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i}, Y_{n-i-i})$ operator is used for states with even priority $p(s)$, then $\text{LPreOdd}_1(i, Y_n, X_n, \ldots, Y_{n-i}, X_{n-i})$ operator is used for states with priority $p(s) + 1$, along with the above priority assignment ensures a correspondence of winning set of the concurrent game and the turn-based game obtained. Also observe that it is taken care that absorbing states of the form $(s, \emptyset, B, 1)$ are assigned an odd priority.

Analogous to the characterization of the limit-winning states in concurrent games by Theorem 1, it can be shown that the set of limit-winning states in $\widetilde{\mathcal{G}}_L$ that intersects with $S$ can be obtained by the $\mu$-calculus expression of Theorem 1 by replacing the $\text{LPreOdd}_1$ and $\text{LPreEven}_1$ operator with $\widetilde{\text{LPreOdd}}_1$ and $\widetilde{\text{LPreEven}}_1$, respectively. Also observe that for the $\mu$-calculus formula characterizing the limit-winning set, for all calls to the $\text{LPreOdd}_1$ and $\text{LPreEven}_1$ operators, the arguments satisfy the inclusion relation assumptions of Lemma 6 and and Lemma 7. The above characterization, the characterization of limit-winning states in concurrent games by $\mu$-calculus expression of Theorem 1 along with Lemma 6 and Lemma 7 yield the following result.

**Theorem 2** *Given a concurrent game structure $\mathcal{G}$ with a parity objective Parity$(p)$, consider the turn-based game structure $\widetilde{\mathcal{G}}_L$ with the objective Parity$(\widetilde{p})$. Let $W_l = \langle\langle 1\rangle\rangle_{limit}(\mathit{Parity}(p))$ in $\mathcal{G}$ and $\widetilde{W}_l = \langle\langle 1\rangle\rangle_{limit}(\mathit{Parity}(\widetilde{p}))$ in $\widetilde{\mathcal{G}}_L$, then $W_l = \widetilde{W}_l \cap S$.*

**Algorithms.** Given a state $s$ let $\beta(s) = 2^{O(|\Gamma_1(s)+\Gamma_2(s)|)}$, and $\beta(S) = \sum_{s\in S} \beta(s)$. The reduction of a concurrent game structure $\mathcal{G}$ to turn-based game structures $\mathcal{G}_L$ for limit-winning satisfy that $|\mathcal{G}_L| = O(\beta(S))$. A turn-based game structure $\mathcal{G}_T$ with a parity objective Parity$(p)$ of $d$-priorities can be reduced to a turn-based deterministic game structure $\mathcal{G}_{\text{TD}}$ with a parity objective Parity$(\widehat{p})$ such that $|\mathcal{G}_{\text{TD}}| = |\mathcal{G}_T| \cdot d$, the priority function $\widehat{p}$ has $d+1$ priorities and the limit-winning states in $\mathcal{G}_T$ can be obtained by solving $\mathcal{G}_{\text{TD}}$. The above reduction of $\mathcal{G}_T$ to $\mathcal{G}_{\text{TD}}$ was established in [4]. Our reduction of concurrent game structures to turn-based game structures (Theorem 2) along with the reduction of [4] makes the rich class of algorithms for turn-based deterministic parity games (such as classical recursive algorithm [15], small-progress measure algorithm [11], strategy improvement algorithm [24], deterministic sub-exponential algorithm [13], solving parity games in big steps [20]) available to compute limit-winning sets for concurrent games with parity objectives. This gives us Theorem 3: the bound for part 2 is obtained by applying the algorithm of [20] and the bound for part 3 is obtained by applying the deterministic sub-exponential algorithm [13].

**Theorem 3** *Given a concurrent game structure $\mathcal{G}$ with a parity objective Parity$(p)$ with $d$ priorities the following assertions hold.*

1. *There is a deterministic turn-based game $\mathcal{G}_{\text{TD}}$ with a parity function $\widetilde{p}$ with $d+1$ priorities such that if $W_l = \langle\langle 1\rangle\rangle_{limit}(\mathit{Parity}(p))$ in $\mathcal{G}$ and $\widetilde{W}_l = \langle\langle 1\rangle\rangle_{limit}(\mathit{Parity}(\widetilde{p}))$ in $\widetilde{\mathcal{G}}_L$, then $W_l = \widetilde{W}_l \cap S$. Moreover, $|\mathcal{G}_{\text{TD}}| = \beta(S) \cdot d$, and the game $\mathcal{G}_{\text{TD}}$ can be constructed in time $O(\beta(S) \cdot d)$.*

2. *The set $\langle\langle 1\rangle\rangle_{limit}(\mathit{Parity}(p))$ can be computed in time $O\big((\beta(S) \cdot d)^{\lfloor \frac{d}{3} \rfloor + 3}\big)$.*

3. *The set* $\langle\langle 1 \rangle\rangle_{limit}\big(Parity(p)\big)$ *can be computed in time* $O\big((\beta(S) \cdot d)^{\sqrt{\beta(S) \cdot d}}\big)$.

The running time of the previous best known algorithm to compute the limit-winning states is $O(|\mathcal{G}|^{2d+2})$ [5, 3]; if $\beta(S)$ is $O(|\mathcal{G}|)$ (which is typically the case since the number of available moves at a state is constant), then Theorem 3 (part 2) yields a better complexity bound. If $\beta(S) = O(|\mathcal{G}|)$ and $d = O(|S|^{1-\varepsilon})$ for $\varepsilon > 0$, then Theorem 3 (part 3) yields a sub-exponential bound to compute the limit-winning states.

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