# A Numerically Stable Procedure for Canonical Coordinates in 2D

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**Abstract**

This note describes a numerically stable procedure to derive a canonical coordinate system for a 2D linear transformation. The determinant of the transformation from the given coordinate system to the canonical coordinate system is 1; thus area is preserved. The transformation is composed of a short sequence of rotations and area-preserving nonuniform scales. If the eigenvectors are real, they map into axes in the canonical coordinate system. If the eigenvectors are complex, the transformation becomes rotationally invariant in the canonical coordinate system. The numerically difficult case is when the transformation is nonsingular, yet the eigenvectors are very nearly colinear.

## 1 Introduction

The characterization of solutions to linear ordinary differential equations in 2D can be accomplished by finding a linear transformation into a so-called canonical coordinate system. This is also called critical point analysis.

Vector fields are often approximated linearly for numerical purposes. The *critical point* of a linear vector field is the (normally unique) point at which the field is zero. Stream lines in a vector field are everywhere tangent to the field, and they represent solutions of the associated differential equation. The qualitative nature of stream lines is determined by classifying the critical point.

Although the theory of classifying critical points is completely known for two and three dimensions (see any text on linear differential equations), in practice there may be numerical issues. When the eigenvalues of the transformation are complex, straightforward application of the theory involves complex computations and gives little intuition. This note describes a procedure that uses only real computations and gives intuition, as well as numerical stability. For omitted background, the reader should consult a text on linear differential equations.

## 2 The Problem

Suppose we are given a coordinate system $(x,y)$ and a nonsingular affine transformation

$$\mathbf{A}\mathbf{p}+\mathbf{c} \qquad \mathbf{p} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{1}$$

where $\mathbf{A}$ is $2 \times 2$ and $\mathbf{c}$ and $\mathbf{p}$ are $2 \times 1$. This expression also defines a 2D vector field. The critical point, where the vector field = 0, is given by

$$\mathbf{p}_{crit} \quad = \quad -\mathbf{A}^{-1}\mathbf{c} \tag{2}$$

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If we make the translation $\mathbf{p}' = \mathbf{p} - \mathbf{p}_{crit}$, then in the coordinates of $\mathbf{p}'$ the critical point is at zero and the transformation is simply $\mathbf{A}\mathbf{p}'$. We assume that this translation has been done as a preprocessing step from now on, and assume that the critical point is at the origin in the $(x,y)$ system and the transformation under consideration is $\mathbf{A}\mathbf{p}$.

Suppose we define a new coordinate system by

$$\mathbf{p} = P\mathbf{q} \qquad \mathbf{q} = \begin{bmatrix} s \\ t \end{bmatrix} \tag{3}$$

where $P$ is $2 \times 2$. Then it is well known that the matrix for the linear transformation that is represented by $\mathbf{A}$ in the given $(x,y)$ system is given by the *similarity transformation*:

$$\mathbf{A}_P = P^{-1}\mathbf{A}P \tag{4}$$

In other words, if $\mathbf{p}_2 = \mathbf{A}\mathbf{p}_1$ and $\mathbf{q}_1 = P\mathbf{p}_1$ and $\mathbf{q}_2 = P\mathbf{p}_2$, then $\mathbf{q}_2 = \mathbf{A}_P\mathbf{q}_1$.

The problem that we address is to find a suitable transformation $P$ with determinant 1 such that the classification of the critical point can be determined by inspection of $\mathbf{A}_P$.

We use the following notation for the elements of $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} a & b \\ e & f \end{bmatrix} \tag{5}$$

### 2.1 Matrices in Canonical Form

In a canonical coordinate system the matrix has one of a few special forms:

$$\begin{bmatrix} a & 0 \\ 0 & f \end{bmatrix} \quad \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad \begin{bmatrix} a & 0 \\ e & a \end{bmatrix} \quad \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \quad \begin{bmatrix} a & b \\ \alpha a & \alpha b \end{bmatrix}$$

Thus the problem is, given a matrix $\mathbf{A}$, find a transformation $P$ such that $P^{-1}\mathbf{A}P$ is in one of these forms.

We included the last case for completeness. The matrix is singular. The vector field has a uniform direction with slope $\alpha$. There are infinitely many critical points, of the form $(-bt, at)$, where $t$ is a real parameter. After a rotation, the problem is essentially one-dimensional. Hereafter we confine our attention to nonsingular cases, in which there is one isolated critical point.

## 3 Area-Preserving Similarity Transformations

For coordinate transformation $P$, a similarity transformation is $P^{-1}\mathbf{A}P$. As mentioned, the transformation preserves area if the determinant of $P$ is 1.

Our main result is that a canonical coordinate system can be derived with a short sequence of area-preserving transformations of two familiar types:

$$\begin{array}{lll} \textit{2D rotation}: & R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} & R^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ \textit{Nonuniform scale}: & S_w = \begin{bmatrix} w & 0 \\ 0 & 1/w \end{bmatrix} & S_w^{-1} = \begin{bmatrix} 1/w & 0 \\ 0 & w \end{bmatrix} \end{array} \tag{6}$$

### 3.1 Invariants of Similarity Transformations

Several invariants of similarity transformations guide us in the composition of a transformation into canonical form. Section 3.2 contains the details of the transformations that will be needed. Recall that the *trace* of $\mathbf{A}$ (tr$\mathbf{A}$) is $a + f$. This is invariant under any similarity transformation.

Let the *rotation matrix* $R$ be given by

$$R = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}, \tag{7}$$

where $c = \cos\theta$, $s = \sin\theta$. Recall that $R^{-1} = R^T$.

The similarity transformation based on $R$ preserves the *difference* between the off-diagonal elements of $\mathbf{A}$.

Let $M = \mathbf{A}_R = R^{-1}\mathbf{A}R$. Then it is possible to choose $c$ and $s$ (i.e., choose $\theta$) so that $m_{11} = m_{22}$, that is, the diagonal elements are equal.

Now we turn our attention to *area-preserving scale* transformations. The similarity transformation of $\mathbf{A}$ based on an area-preserving scale preserves the diagonal elements $a$ and $f$, and preserves the product $be$. It may be chosen to cause the off-diagonal elements of the new matrix to have equal magnitudes.

## 3.2 Details

The results stated in this section can be obtained by standard methods of linear algebra and trigonometry. Derivations are omitted.

With the notation of the previous section (Eq. 7), and $\sqrt{\phantom{x}}$ denoting the positive square root, let $M = \mathbf{A}_R = R^{-1}\mathbf{A}R$. We define these quantities:

$$D = (f-a)^2 + (b+e)^2 \tag{8}$$

$$c = \begin{cases} -\sqrt{\frac{1}{2} + \frac{b+e}{2\sqrt{D}}} & \text{if } a > f \text{ and } (b+e) \leq 0 \\ \sqrt{\frac{1}{2} + \frac{b+e}{2\sqrt{D}}} & \text{otherwise.} \end{cases} \tag{9}$$

$$s = \begin{cases} -\sqrt{\frac{1}{2} - \frac{b+e}{2\sqrt{D}}} & \text{if } a > f \text{ and } (b+e) > 0 \\ \sqrt{\frac{1}{2} - \frac{b+e}{2\sqrt{D}}} & \text{otherwise.} \end{cases} \tag{10}$$

The expression $(b+e)/(2\sqrt{D})$ is evaluated as 0 when $(b+e) = 0$, even if $(f-a)$ is also 0. The $R$ defined by these values of $c$ and $s$ produces the following value for $M$:

$$M = \begin{bmatrix} \frac{1}{2}(a+f) & \frac{1}{2}\sqrt{D} + \frac{1}{2}b - \frac{1}{2}e \\ \frac{1}{2}\sqrt{D} + \frac{1}{2}e - \frac{1}{2}b & \frac{1}{2}(a+f) \end{bmatrix} \tag{11}$$

Thus the diagonal elements have been equalized.

The reason for choosing the signs of square roots as specified in Equations 9 and 10 is to ensure stability in the values of the elements of $M$ when $(f-a)$ and/or $(b+e)$ are close to 0. That is, a small change in one of these quantities that causes its sign to change should not cause a discontinuous change in $M$.

To choose the nonuniform scale $S_w$ that equalizes the magnitudes of the off-diagonal elements, we note that

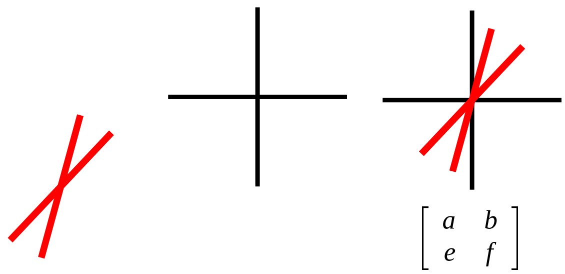
$$S_w^{-1} \begin{bmatrix} a & b \\ e & f \end{bmatrix} S_w = \begin{bmatrix} a & b/w^2 \\ ew^2 & f \end{bmatrix} \tag{12}$$

so $w = \sqrt[4]{|b/e|}$ is the desired scale factor, provided that neither $b$ nor $e$ is zero or "too close to zero." This is discussed further in Section 5.

# 4 General Procedure

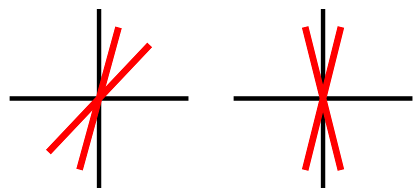
At a high level the general procedure to construct the canonical coordinate system is the following:

1. Translate critical point to the origin (assumed done beforehand).



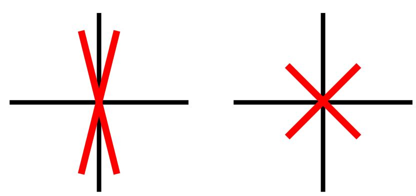
$$\begin{bmatrix} a & b \\ e & f \end{bmatrix}$$

2. Apply a rotation that equalizes the diagonal elements, per Eq. 11.



$$\begin{bmatrix} a & b \\ e & f \end{bmatrix} \qquad \begin{bmatrix} a' & b' \\ e' & a' \end{bmatrix}$$

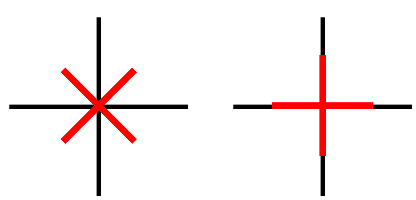
3. Apply an area-preserving scale to equalize magnitudes of off-diagonal elements, per Eq. 12.



$$\begin{bmatrix} a & b \\ e & a \end{bmatrix} \qquad \begin{bmatrix} a & b' \\ \pm b' & a \end{bmatrix}$$

   If the off-diagonal elements have opposite signs, this is a canonical form; otherwise, continue with step 4.

4. If the off-diagonal elements have the same sign (i.e., the matrix is symmetric), rotate by $-45$ degrees $(-\pi/4)$.



$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \qquad \begin{bmatrix} a' & 0 \\ 0 & f' \end{bmatrix}$$

The coordinate transformation $P$ is the left-to-right composition of the transformations in the steps, $P = R_\theta S_w$ or $P = R_\theta S_w R_{-\pi/4}$, where $R_\theta$ is given by Equations 8–10. The case in which step 3 is ill-defined is discussed in Section 5; for this case, simply $P = R_\theta$.

### 4.1 Streamline Characteristics and the Canonical Matrix

Let us look at the case in which the off-diagonal elements have opposite signs after step 3. Note that the matrix can be rewritten as the product of a uniform scale and a rotation. Thus this matrix is invariant under any rotation of the coordinate system. The streamlines are spirals.

- If $a < 0$, they spiral in toward the origin.
- If $a > 0$, they spiral out from the origin.
- If $a = 0$, streamlines are circles.

Whether the spirals are clockwise or counter-clockwise depends on the signs of $a$ and $b$.

Now suppose the off-diagonal elements have the same sign after step 3, and let's look at the situation after step 4.

- If $af < 0$, the streamlines are hyperbolic.
- If $a < 0$ and $f < 0$, all streamlines proceed in a straight line toward the origin.
- If $a > 0$ and $f > 0$, all streamlines proceed in a straight line away from the origin.

There are two distinct real eigenvectors, which align with the axes.

Finally, suppose that the matrix $\mathbf{A}$ is singular or nearly singular. With the aid of the Schwartz inequality it can be shown that the off-diagonal elements have the same sign after step 2, and therefore after step 3. After step 4, either $a'$ or $f'$ will be 0, within numerical accuracy.

### 4.2 An Example

Suppose we start with the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} a & b \\ e & f \end{bmatrix} \tag{13}$$

Note that the determinant is $\det(\mathbf{A}) = -5$. We have $(f-a) = 4$, $(b+e) = 3$, $D = 25$, $\sqrt{D} = 5$. By Equations 9 and 10, $c = \sqrt{0.8}$ and $s = \sqrt{0.2}$. These values determine the matrix $R$ by Eq. 7. After the similarity transformation by $R$, according to Eq. 11:

$$\mathbf{A}_R = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \tag{14}$$

For the next step, choose $w = \sqrt[4]{3/2}$ and apply the similarity transformation by $S_w$ (see Equations 6 and 12):

$$\mathbf{A}_{RS_w} = \begin{bmatrix} 1 & \sqrt{6} \\ \sqrt{6} & 1 \end{bmatrix} \tag{15}$$

The final step, since the matrix is symmetric, is to apply a rotation of $-\pi/4$ as a similarity transformation. Thus the composite coordinate transformation is

$$P = RS_wR_{-\pi/4} = \begin{bmatrix} \sqrt{\frac{4}{5}} & -\sqrt{\frac{1}{5}} \\ \sqrt{\frac{1}{5}} & \sqrt{\frac{4}{5}} \end{bmatrix} \begin{bmatrix} \sqrt[4]{\frac{3}{2}} & 0 \\ 0 & \sqrt[4]{\frac{2}{3}} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{bmatrix} \tag{16}$$

$$= \frac{\sqrt[4]{6}}{\sqrt{60}} \begin{bmatrix} 2\sqrt{3}+\sqrt{2} & 2\sqrt{3}-\sqrt{2} \\ \sqrt{3}-2\sqrt{2} & \sqrt{3}+2\sqrt{2} \end{bmatrix} \approx \begin{bmatrix} 0.986 & 0.414 \\ -0.222 & 0.921 \end{bmatrix} \tag{17}$$

giving

$$\mathbf{A}_P = \begin{bmatrix} 1-\sqrt{6} & 0 \\ 0 & 1+\sqrt{6} \end{bmatrix} \tag{18}$$

We can easily verify that $\det(\mathbf{A}_P) = -5$, as it should for any similarity transformation. More importantly, $\det(P) = 1$.

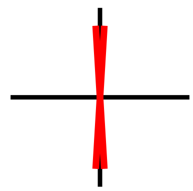
Since the canonical matrix is diagonal, the eigenvalues and eigenvectors of $\mathbf{A}$ are real. The diagonal elements of $\mathbf{A}_P$ define the eigenvalues. The columns of $P$ define the eigenvectors. Since $\mathbf{A}$ is asymmetric, the eigenvectors are not orthogonal. Also, they are not of unit length because we desired $\det(P) = 1$. The eigenvectors map into the axes of the canonical coordinate system.

The vector field is hyperbolic because the eigenvalues have opposite signs. The streamlines' asymptotic directions are given by the eigenvectors. They approach the origin from the direction of the eigenvector whose eigenvalue is negative, $\binom{0.986}{-0.222}$ in this case, and from the opposite direction. They approach infinity in the direction of the eigenvector whose eigenvalue is positive, which is $\binom{0.414}{0.921}$, and in the opposite direction.

# 5 Numerically Unstable Case

The only problematical case occurs when the matrix is asymmetric after the diagonal elements are equalized in step 2 of the general procedure, and one of the off-diagonal elements is nearly zero. For example,

$$\begin{bmatrix} a & \pm\varepsilon \\ b & a \end{bmatrix}$$



An extremely uneven nonuniform scale would be needed in step 3 of the general procedure. Depending on whether the sign of $\varepsilon$ is the same as that of $b$ or opposite, the eigenvalues are real or complex. This condition might be determined by rounding accidents of numerical operations.

If $\varepsilon$ is exactly zero, the matrix is already in a canonical form, and no transformation is appropriate. We recommend a practical treatment as follows: When $|\varepsilon| \ll |b|$, do not rescale, and treat $\varepsilon$ as 0.

A suitable threshold ratio depends on the accuracy of the data. For example, if there are about four significant digits of accuracy, then the criterion might be $|\varepsilon| < 10^{-4}|b|$. This cut-off would mean that the scale value $w$ in step 3 of the general procedure would never exceed 10 or fall below 1/10 (see Eq. 12).

With one off-diagonal element being 0, the streamlines of the associated vector field behave like "defective" spirals. There is only one distinct eigenvector: $(0,1)$ in the case in which the upper off-diagonal element is 0, and $(1,0)$ when the lower off-diagonal element is 0. Thus streamlines are all asymptotically parallel to the $y$-axis in the first case and to the $x$-axis in the second case.

# 6 Conclusion

We have presented a "cookbook" procedure for finding a 2D coordinate transformation that puts a given matrix into a canonical form useful for critical point classification. It would be useful to extend the technique to three dimensions.