On the Measured Equation of Invariance

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ABSTRACT

The key to the method of measured equation of invariance (MEI) is the postulate: "the MEI is invariant to the excitation". In this paper, we proved that the MEI is independent of the excitation with the error bounded by $O(h^2)$, where h is the discretization step. We also proved that the consistent condition $||L\phi - M\phi/h^2|| = \varepsilon$, where L is the partial differential operator, and M the equivalent MEI operator. If the MEI coefficients C_i^* $(i = 1, \ldots, 4)$ are determined from the special distribution named metrons on the boundary of the object, then $\varepsilon = \delta J(s)$, otherwise $\varepsilon = J(s)$, where $J(s)$ is a functional of the source distribution s, and $\delta J(s)$ is the perturbation of the functional $J(s)$. And finally we pointed out that the error between the accurate solution and the solution of MEI is bounded by two terms, one is caused by the FD approximation at interior nodes which is proportional to h^2, another is caused by MEI and is independent of h but proportional to $\delta J(s)$ which ensured the accuracy of MEI solution.

Keywords: Measured Equation of Invariance (MEI), postulates, error bound, stability, consistency condition, convergence property

1 Introduction

Measured Equation of Invariance(MEI) is a new concept in computational electromagnetics[MPCL92]. MEI is used to derive the local finite difference (FD) like equation at mesh boundary where the conventional FD/FEM approach fails. It is demonstrated that the MEI technique can be used to terminate the meshes very close to the object boundary and still strictly preserves the sparsity of the FD equations. Therefore, the final system matrix encountered by MEI is a sparse matrix with size similar to that of integral equation methods, which results in dramatic savings in computing time and memory usage compared to other known methods. It has been successfully used to analyze electromagnetic scattering problems [MPCL92, HLM94, HM94, CHZ96, CHed], and microwave integrated circuits [PPM93a, PPM93b]. MEI concept has also been applied to IC interconnect parasitic extraction [HSD96] [SHD96], which is becoming more and more important with the increase of clock rate and decrease of feature size.

Recently, however, some papers[JL94] [JL95a] propose some doubts on the third postulation of the MEI coefficients: invariant to excitations, which is one of the main topics of this paper. It then results in some arguments [ML95, JL95b, Mei95, JL95c]. In fact, there is a mistake in the derivation in paper [JL95a] as pointed by Mei [Mei95]. In their final operators (see Eq.9 and 10 in [JL95a]), they included the term $(k\,b)^{-2}$ which is just the source of the difference of two different operators, but they omitted the term $(k\,b)^{-1}$ in the derivation of the final operator. Although they have some doubts on MEI, they still admit in the papers that MEI is an efficient technique for the truncation of mesh boundaries [JL94].

In this paper, we provided solid proof and reasonable comments on the method of MEI.

2 Basic MEI idea

Considering the EM scattering problem of a general cylinder (not necessary conducting, may be penetrable media [HLM94]) and several layers of 2D mesh around the cross-section of the cylinder shown in Fig.1. For the sake of convenience and without loss of generality, let the horizontal and vertical discretization step size the same and denoted as h.

At the interior nodes of the mesh, the following 5-points finite difference equation (FD)

$$\sum_{i=0}^{4} c_i \phi(\bar{r}_i) = O(h^4) \tag{1}$$

can be applied, where in Cartesian coordinate system and under uniform media assumption, $c_1 = c_2 = c_3 = c_4 = 1, c_0 = (kh)^2 - 4$, and $\bar{r}_i$ is the position vector of the ith node.

On the truncated boundary nodes as shown in Fig.1, a different type of relation has to be applied, such as traditional E.W., M.W., and ABC's. Mei [MPCL92] postulated the existence of the following linear equation for the truncated boundary nodes

$$\sum_{i=0}^{3} c_i \phi(\bar{r}_i) = 0 \tag{2}$$

Figure 1: A general cylinder and its 2D mesh scheme

which is called MEI (Measured Equation of Invariance), and $c_i, i = 0, 1, 2, 3$ are called MEI coefficients, which by Mei's postulation have three properties (i) location dependent, (ii) geometric specific, (iii) invariant to the excitation. Among them, the third one, invariance to excitation is the base of the method of MEI.

3 Theorems on MEI

Assume that: potential or field function ϕ in an open region outside an object satisfies the partial differential equation (PDE) $L\phi = 0$, where L is a partial differential operator, and ϕ may be expressed as $\phi(\bar{r}) = \int_\Gamma s(\bar{r}')G(\bar{r}, \bar{r}')d\Gamma' = \pounds[s(\bar{r}')]$, where $\bar{r}, \bar{r}'$ stand for field and source points respectively, Γ for the boundary of the object, s for the source distribution, G and $\pounds$ for Green's function and integral operator.

First we will prove the theorem that states *the MEI coefficients are independent of source distributions with error bounded by* $O(h^2)$.

The proof begins with definitions of some concepts.

Definition 1. Let C denote the continuous function space consisting of the continuous functions defined on the boundary Γ of the cylinder.

For any excitation, the induced source distribution $s(l)$ (l is the length along the boundary Γ) is always a continuous function, so $s(l) \in C$.

The potential function $\phi(\bar{r})$ should satisfy Laplace equation for static problems and Helmholtz equation for dynamic problems, so its derivatives of the second order should be continuous. According to the property of the Green's function $G(\bar{r}, \bar{r}')$, in fact, even higher order derivatives of the potential function $\phi(\bar{r})$ are continuous.

Defining vector $\bar{\phi} = (\phi(\bar{r}_1), \phi(\bar{r}_2), \phi(\bar{r}_3), \phi(\bar{r}_0))$, here $\bar{r}_i$ is the position vector of the ith node of MEI, then

Definition 2. Let $\Phi = \{\bar{\phi}, \phi(\bar{r}) = \pounds[s(l)], \forall s(l) \in C\}$ be the vector space consisting of vectors $\bar{\phi}$ produced by all source distributions.

Definition 3. $\Psi = \{\ \bar{c} = (c_1, c_2, c_3, 1), c_i$ are any bounded complex numbers$\}$ be the space of MEI coefficients vectors, here the MEI coefficient c_0 has been normalized to 1.

Definition 4. If $\bar{c} \cdot \bar{\phi} = \sum_{i=0}^{3} c_i \phi(\bar{r}_i) = 0$, then $\bar{c}$ is perpendicular to $\bar{\phi}$, denoted as $\bar{c} - \bar{\phi}$. If $\forall \bar{\phi} \in \Phi, \bar{c} - \bar{\phi}$, then $\bar{c}$ is perpendicular to the space Φ, denoted as $\bar{c} - \Phi$.

If $\bar{c} - \Phi$, we say $\bar{c}$ is independent of Φ. Since $\bar{c} \cdot \bar{\phi} = 0, \forall \bar{\phi} \in \Phi \Longleftrightarrow \bar{c} - \Phi$, so if we want to prove $\bar{c}$ is independent of space Φ, we only need to prove that $\forall \bar{\phi} \in \Phi, \bar{c} \cdot \bar{\phi} = 0$.

However, in numerical analysis, we usually have $|\bar{c} \cdot \bar{\phi}| \leq \varepsilon \neq 0, \forall \bar{\phi} \in \Phi$, here ε is generally a small quantity. In this case obviously, $\bar{c}$ is not independent of space Φ, but we can say $\bar{c}$ is independent of space Φ with error bounded by ε.

For example, the FD equation (1) may be rewritten as $\bar{c} \cdot \bar{\phi} = O(h^4)$, so we can say the coefficient vector $\bar{c} = (k^2h^2 - 4, 1, 1, 1, 1)$ of the FD equation is independent of the space Φ with error bounded by $O(h^4)$.

Let $p = \bar{c} \cdot \bar{\phi}$ as the projection from $\bar{c}$ to $\bar{\phi}$, then p describes the coherence between $\bar{c}$ and $\bar{\phi}$. The smaller the projection p, the weaker the coherence between $\bar{c}$ and $\bar{\phi}$.

We can rewrite the MEI Eq.2 as

$$\bar{c} \cdot \bar{\phi} = 0, \quad or \quad \bar{c} - \bar{\phi}, \ \ \forall \bar{\phi} \in \Phi \tag{3}$$

which means $\bar{c}$ is rigorously independent of the equivalent source distribution on the surface of the scattering cylinder, or independent of incident field because the current distribution is actually induced by the incident field.

As mentioned above, the invariance to excitation is only a postulation or guess. Is there really a MEI coefficients vector $\bar{c}$ that is rigorously independent of the excitationΓ The question is answered by the following theorem.

Theorem 1. Assume a vector $\bar{c}^* \in \Psi$ is perpendicular to three linear independent vectors of the space Φ, i.e.

$$\bar{c}^* \cdot \bar{\phi}_m = \sum_{i=0}^{3} c_i^* \phi_m(\bar{r}_i) = 0, \quad m = 1, 2, 3 \quad \bar{\phi}_m \in \Phi \tag{4}$$

then,

$$\bar{c}^* \cdot \bar{\phi} = \sum_{i=0}^{3} c_i^* \phi(\bar{r}_i) = O(h^2) \quad \forall \phi \in \Phi \tag{5}$$

which means the MEI coefficient vector $\bar{c}^*$ is independent of excitation with error bounded by $O(h^2)$.

Proof: $\forall\bar{\phi}\in\Phi$, define the projection from $\bar{c}^*$ to $\bar{\phi}$ as

$$p=\bar{c}^*\cdot\bar{\phi}=\sum_{i=0}^{3}c_i^*\phi(\bar{r}_i) \tag{6}$$

The condition Eq.4 is just a system of linear algebraic equation with respect to the MEI coefficients c_1^*, c_2^*, and c_3^*

$$\begin{bmatrix}\phi_1(\bar{r}_1) & \phi_1(\bar{r}_2) & \phi_1(\bar{r}_3)\\ \phi_2(\bar{r}_1) & \phi_2(\bar{r}_2) & \phi_2(\bar{r}_3)\\ \phi_3(\bar{r}_1) & \phi_3(\bar{r}_2) & \phi_3(\bar{r}_3)\end{bmatrix}\begin{bmatrix}c_1^*\\ c_2^*\\ c_3^*\end{bmatrix}=-\begin{bmatrix}\phi_1(\bar{r}_0)\\ \phi_2(\bar{r}_0)\\ \phi_3(\bar{r}_0)\end{bmatrix} \tag{7}$$

whose solution can be easily expressed according to Gramm's rule

$$c_i^*=\frac{D_i}{D}\qquad i=1,2,3 \tag{8}$$

where

$$D=\begin{vmatrix}\phi_1(\bar{r}_1) & \phi_1(\bar{r}_2) & \phi_1(\bar{r}_3)\\ \phi_2(\bar{r}_1) & \phi_2(\bar{r}_2) & \phi_2(\bar{r}_3)\\ \phi_3(\bar{r}_1) & \phi_3(\bar{r}_2) & \phi_3(\bar{r}_3)\end{vmatrix} \tag{9}$$

$$D_1=-\begin{vmatrix}\phi_1(\bar{r}_0) & \phi_1(\bar{r}_2) & \phi_1(\bar{r}_3)\\ \phi_2(\bar{r}_0) & \phi_2(\bar{r}_2) & \phi_2(\bar{r}_3)\\ \phi_3(\bar{r}_0) & \phi_3(\bar{r}_2) & \phi_3(\bar{r}_3)\end{vmatrix} \tag{10}$$

$$D_2=-\begin{vmatrix}\phi_1(\bar{r}_1) & \phi_1(\bar{r}_0) & \phi_1(\bar{r}_3)\\ \phi_2(\bar{r}_1) & \phi_2(\bar{r}_0) & \phi_2(\bar{r}_3)\\ \phi_3(\bar{r}_1) & \phi_3(\bar{r}_0) & \phi_3(\bar{r}_3)\end{vmatrix} \tag{11}$$

$$D_3=-\begin{vmatrix}\phi_1(\bar{r}_1) & \phi_1(\bar{r}_2) & \phi_1(\bar{r}_0)\\ \phi_2(\bar{r}_1) & \phi_2(\bar{r}_2) & \phi_2(\bar{r}_0)\\ \phi_3(\bar{r}_1) & \phi_3(\bar{r}_2) & \phi_3(\bar{r}_0)\end{vmatrix} \tag{12}$$

Therefore, the projection p can be expressed as

$$\begin{aligned}p &= \phi(\bar{r}_1)c_1^*+\phi(\bar{r}_2)c_2^*+\phi(\bar{r}_3)c_3^*+\phi(\bar{r}_0)\\ &= \frac{\phi(\bar{r}_1)D_1+\phi(\bar{r}_2)D_2+\phi(\bar{r}_3)D_3+\phi(\bar{r}_0)D}{D}\\ &= \frac{D_4}{D}\end{aligned} \tag{13}$$

where

$$D_4=-\begin{vmatrix}\phi(\bar{r}_1) & \phi(\bar{r}_2) & \phi(\bar{r}_3) & \phi(\bar{r}_0)\\ \phi_1(\bar{r}_1) & \phi_1(\bar{r}_2) & \phi_1(\bar{r}_3) & \phi_1(\bar{r}_0)\\ \phi_2(\bar{r}_1) & \phi_2(\bar{r}_2) & \phi_2(\bar{r}_3) & \phi_2(\bar{r}_0)\\ \phi_3(\bar{r}_1) & \phi_3(\bar{r}_2) & \phi_3(\bar{r}_3) & \phi_3(\bar{r}_0)\end{vmatrix}$$

$$
\begin{aligned}
&= -\left|\begin{array}{ll}
\phi(\bar r_0) - h\phi'_t(\bar r_0) + 0.5h^2\phi''_t(\bar r_0) + O(h^3) & \phi(\bar r_0) - h\phi'_n(\bar r_0) + O(h^2) \\
\phi_1(\bar r_0) - h\phi'_{t1}(\bar r_0) + 0.5h^2\phi''_{t1}(\bar r_0) + O(h^3) & \phi_1(\bar r_0) - h\phi'_{n1}(\bar r_0) + O(h^2) \\
\phi_2(\bar r_0) - h\phi'_{t2}(\bar r_0) + 0.5h^2\phi''_{t2}(\bar r_0) + O(h^3) & \phi_2(\bar r_0) - h\phi'_{n2}(\bar r_0) + O(h^2) \\
\phi_3(\bar r_0) - h\phi'_{t3}(\bar r_0) + 0.5h^2\phi''_{t3}(\bar r_0) + O(h^3) & \phi_3(\bar r_0) - h\phi'_{n3}(\bar r_0) + O(h^2)
\end{array}\right. \\
&\qquad \left.\begin{array}{ll}
\phi(\bar r_0) + h\phi'_t(\bar r_0) + 0.5h^2\phi''_t(\bar r_0) + O(h^3) & \phi(\bar r_0) \\
\phi_1(\bar r) + h\phi'_{t1}(\bar r_0) + 0.5h^2\phi''_{t1}(\bar r_0) + O(h^3) & \phi_1(\bar r_0) \\
\phi_2(\bar r) + h\phi'_{t2}(\bar r_0) + 0.5h^2\phi''_{t2}(\bar r_0) + O(h^3) & \phi_2(\bar r_0) \\
\phi_3(\bar r) + h\phi'_{t3}(\bar r_0) + 0.5h^2\phi''_{t3}(\bar r_0) + O(h^3) & \phi_3(\bar r_0)
\end{array}\right| \\
&= -\left|\begin{array}{ll}
-h\phi'_t(\bar r_0) + 0.5h^2\phi''_t(\bar r_0) + O(h^3) & -h\phi'_n(\bar r_0) + O(h^2) \\
-h\phi'_{t1}(\bar r_0) + 0.5h^2\phi''_{t1}(\bar r_0) + O(h^3) & -h\phi'_{n1}(\bar r_0) + O(h^2) \\
-h\phi'_{t2}(\bar r_0) + 0.5h^2\phi''_{t2}(\bar r_0) + O(h^3) & -h\phi'_{n2}(\bar r_0) + O(h^2) \\
-h\phi'_{t3}(\bar r_0) + 0.5h^2\phi''_{t3}(\bar r_0) + O(h^3) & -h\phi'_{n3}(\bar r_0) + O(h^2)
\end{array}\right. \\
&\qquad \left.\begin{array}{ll}
h\phi'_t(\bar r_0) + 0.5h^2\phi''_t(\bar r_0) + O(h^3) & \phi(\bar r_0) \\
h\phi'_{t1}(\bar r_0) + 0.5h^2\phi''_{t1}(\bar r_0) + O(h^3) & \phi_1(\bar r_0) \\
h\phi'_{t2}(\bar r_0) + 0.5h^2\phi''_{t2}(\bar r_0) + O(h^3) & \phi_2(\bar r_0) \\
h\phi'_{t3}(\bar r_0) + 0.5h^2\phi''_{t3}(\bar r_0) + O(h^3) & \phi_3(\bar r_0)
\end{array}\right| \\
&= h^4 \left|\begin{array}{llll}
\phi'_t(\bar r_0) & \phi'_n(\bar r_0) & \phi''_t(\bar r_0) & \phi(\bar r_0) \\
\phi'_{t1}(\bar r_0) & \phi'_{n1}(\bar r_0) & \phi''_{t1}(\bar r_0) & \phi_1(\bar r_0) \\
\phi'_{t2}(\bar r_0) & \phi'_{n2}(\bar r_0) & \phi''_{t2}(\bar r_0) & \phi_2(\bar r_0) \\
\phi'_{t3}(\bar r_0) & \phi'_{n3}(\bar r_0) & \phi''_{t3}(\bar r_0) & \phi_3(\bar r_0)
\end{array}\right| + O(h^5) \qquad (14)
\end{aligned}
$$

where $\phi'_{ti}(\bar r_0) = \frac{\partial}{\partial s}\phi_i(\bar r_0)$, $\phi'_{ni}(\bar r_0) = \frac{\partial}{\partial n}\phi_i(\bar r_0)$, and s and n are the two orthogonal directions as shown in Fig. 1. Similarly,

$$
\begin{aligned}
D &= -\left|\begin{array}{lll}
\phi_1(\bar r_1) & \phi_1(\bar r_2) & \phi_1(\bar r_3) \\
\phi_2(\bar r_1) & \phi_2(\bar r_2) & \phi_2(\bar r_3) \\
\phi_3(\bar r_1) & \phi_3(\bar r_2) & \phi_3(\bar r_3)
\end{array}\right| \\
&= -\left|\begin{array}{lll}
\phi_1(\bar r_0) - h\phi'_{t1}(\bar r_0) + O(h^2) & \phi_1(\bar r_0) - h\phi'_{n1}(\bar r_0) + O(h^2) & \phi_1(\bar r_0) + h\phi'_{t1}(\bar r_0) + O(h^2) \\
\phi_2(\bar r_0) - h\phi'_{t2}(\bar r_0) + O(h^2) & \phi_2(\bar r_0) - h\phi'_{n2}(\bar r_0) + O(h^2) & \phi_2(\bar r_0) + h\phi'_{t2}(\bar r_0) + O(h^2) \\
\phi_3(\bar r_0) - h\phi'_{t3}(\bar r_0) + O(h^2) & \phi_3(\bar r_0) - h\phi'_{n3}(\bar r_0) + O(h^2) & \phi_3(\bar r_0) + h\phi'_{t3}(\bar r_0) + O(h^2)
\end{array}\right| \\
&= 2h^2 \left|\begin{array}{lll}
\phi'_{t1}(\bar r_0) & \phi'_{n1}(\bar r_0) & \phi_1(\bar r_0) \\
\phi'_{t2}(\bar r_0) & \phi'_{n2}(\bar r_0) & \phi_2(\bar r_0) \\
\phi'_{t3}(\bar r_0) & \phi'_{n3}(\bar r_0) & \phi_3(\bar r_0)
\end{array}\right| + O(h^3) \qquad (15)
\end{aligned}
$$

Therefore, the projection

$$
p = \frac{h^2}{2} \frac{\left|\begin{array}{llll}
\phi'_t(\bar r_0) & \phi'_n(\bar r_0) & \phi''_t(\bar r_0) & \phi(\bar r_0) \\
\phi'_{t1}(\bar r_0) & \phi'_{n1}(\bar r_0) & \phi''_{t1}(\bar r_0) & \phi_1(\bar r_0) \\
\phi'_{t2}(\bar r_0) & \phi'_{n2}(\bar r_0) & \phi''_{t2}(\bar r_0) & \phi_2(\bar r_0) \\
\phi'_{t3}(\bar r_0) & \phi'_{n3}(\bar r_0) & \phi''_{t3}(\bar r_0) & \phi_3(\bar r_0)
\end{array}\right|}{\left|\begin{array}{lll}
\phi'_{t1}(\bar r_0) & \phi'_{n1}(\bar r_0) & \phi_1(\bar r_0) \\
\phi'_{t2}(\bar r_0) & \phi'_{n2}(\bar r_0) & \phi_2(\bar r_0) \\
\phi'_{t3}(\bar r_0) & \phi'_{n3}(\bar r_0) & \phi_3(\bar r_0)
\end{array}\right|} = \varepsilon h^2 \qquad (16)
$$

Figure 2: Cylinder with a point charge

where ε is independent of h. The theorem is proved.

It should be noted that in the proof, if $\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3 \notin \Phi$, the conclusion is still right, which means the MEI with $O(h^2)$ residue is not unique, or there are infinite sets of MEI coefficients that are independent of the excitation with error bounded by $O(h^2)$. However, it will be shown in next section that the residue of the consistent condition will be minimized if the MEI coefficients are determined by the *metrons* on the surface of the cylinder, i.e., $\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3 \in \Phi$. This is necessary to ensure the accuracy of MEI solution.

Now let's use a simple 2D electro-static example to verify the theorem 1. The structure is shown in Fig. 2, which is a perfect conducting circular cylinder with a point charge q outside it at a distance d. The radius of the cylinder is a. We have analytical solution for the field distribution outside the cylinder. The outside field distribution is equivalent to the field generated by q and its image charge $-q$ placed at $p = a^2/d$ in free space. Let ϕ^{in} be the field generated by the original charge q, ϕ^s be the field generated by induced charge on the surface of the cylinder due to q, and ϕ^t be the total and actual field. Note that ϕ^s is actually equivalent to the field generated by the image charge $-q$. We have the exact solutions

$$\phi^{in} = -\frac{\varepsilon_0 q}{4\pi} ln(r^2 - 2rdcos\theta + d^2) \tag{17}$$

$$\phi^s = \frac{\varepsilon_0 q}{4\pi} ln(r^2 - 2rpcos\theta + p^2) \tag{18}$$

$$\phi^t = \phi^{in} + \phi^s = \frac{\varepsilon_0 q}{4\pi} ln\left(\frac{r^2 - 2rpcos\theta + p^2}{r^2 - 2rdcos\theta + d^2}\right) \tag{19}$$

To solve this problem numerically, we need to first make circular meshes around the cylinder as shown in Fig. 3. Here for the sake of simplicity, only uniform meshes are adopted. We actually only need to get the field distribution (referred to as "scattering field") generated

Figure 3: 2D cylinder mesh

by the charges induced by q, then by adding ϕ^{in} we can get the total field ϕ^t. The scattering field satisfies Laplace equation in circular coordinates,

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0 \tag{20}$$

while the boundary condition on the surface of the cylinder for the scattering field is

$$\phi^s = -\phi^{in} = \frac{\varepsilon_0 q}{4\pi} ln(a^2 - 2adcos\theta + d^2) \tag{21}$$

Let h_r and h_θ be the step size in the **r** and θ directions. Using central difference to approximate differential, we have FD relations for interior nodes as shown in Fig. 3

$$\sum_{i=0}^{4} c_i \phi(\bar{r}_i) = 0 \tag{22}$$

$$\begin{cases} c_1 = 1 + \frac{h_r}{2r} \\ c_4 = 1 - \frac{h_r}{2r} \\ c_2 = c_3 = (\frac{h_r}{h_\theta r})^2 \\ c_0 = -2[1 + (\frac{h_r}{h_\theta r})^2] \end{cases} \tag{23}$$

For the boundary nodes, we use three metrons in the MEI procedure in section 2 to get the MEI coefficients. We have three sets of metrons, (1) set A: on the surface of the cylinder, i.e., $1, cos(\pi l/L), sin(\pi l/L)$, where L is the perimeter of the cylinder, and l is the length of the arc from some starting point; (2) set B: just three point charge with unit charge, they are put equal distance on the circle with radius $a/2$ with the same origin as the original cylinder; (3) set C: even more general, three metrons put on the y-axis with equal distance.

After obtaining the MEI coefficients, we use the actual field distribution ϕ^s as in Eq. 19 to test the residue of the MEI equations with respect to the refinement times Fig. 4 to 6. In the figures, we set radius to be $a = 20$, the distance $d = 60$. We refine the mesh uniformly each time in both directions and keep mesh distance at boundaries the same all the time. It can be seen that the residue of MEI equation decrease with refinement times (and hence step size) quadratically (on the order of h^2) for all three sets of metrons, even though set B and set C are not on the surface of the cylinder. However, not all the three set metrons give good accuracy of the solution. The reason will be seen in the following section.

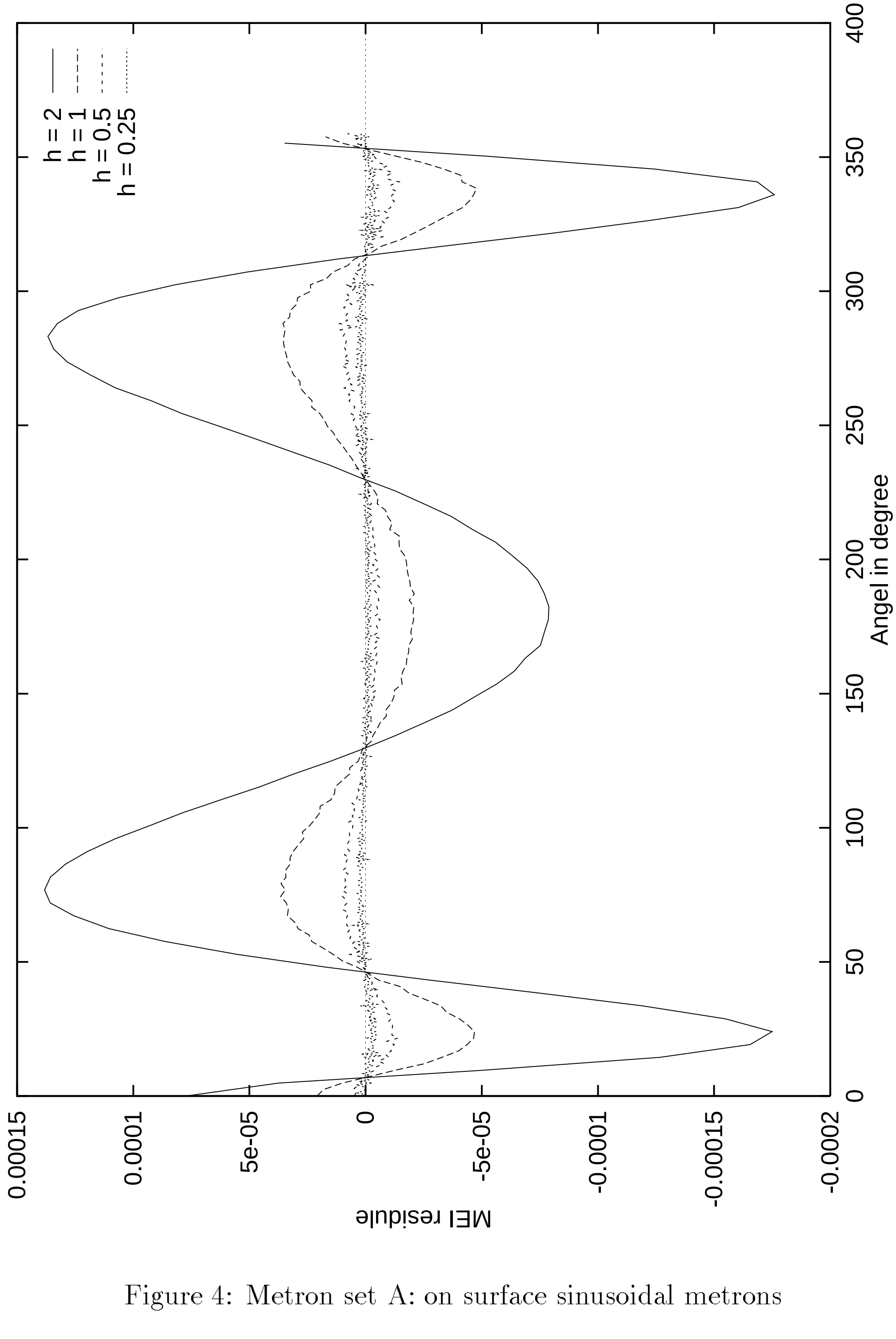

Figure 4: Metron set A: on surface sinusoidal metrons

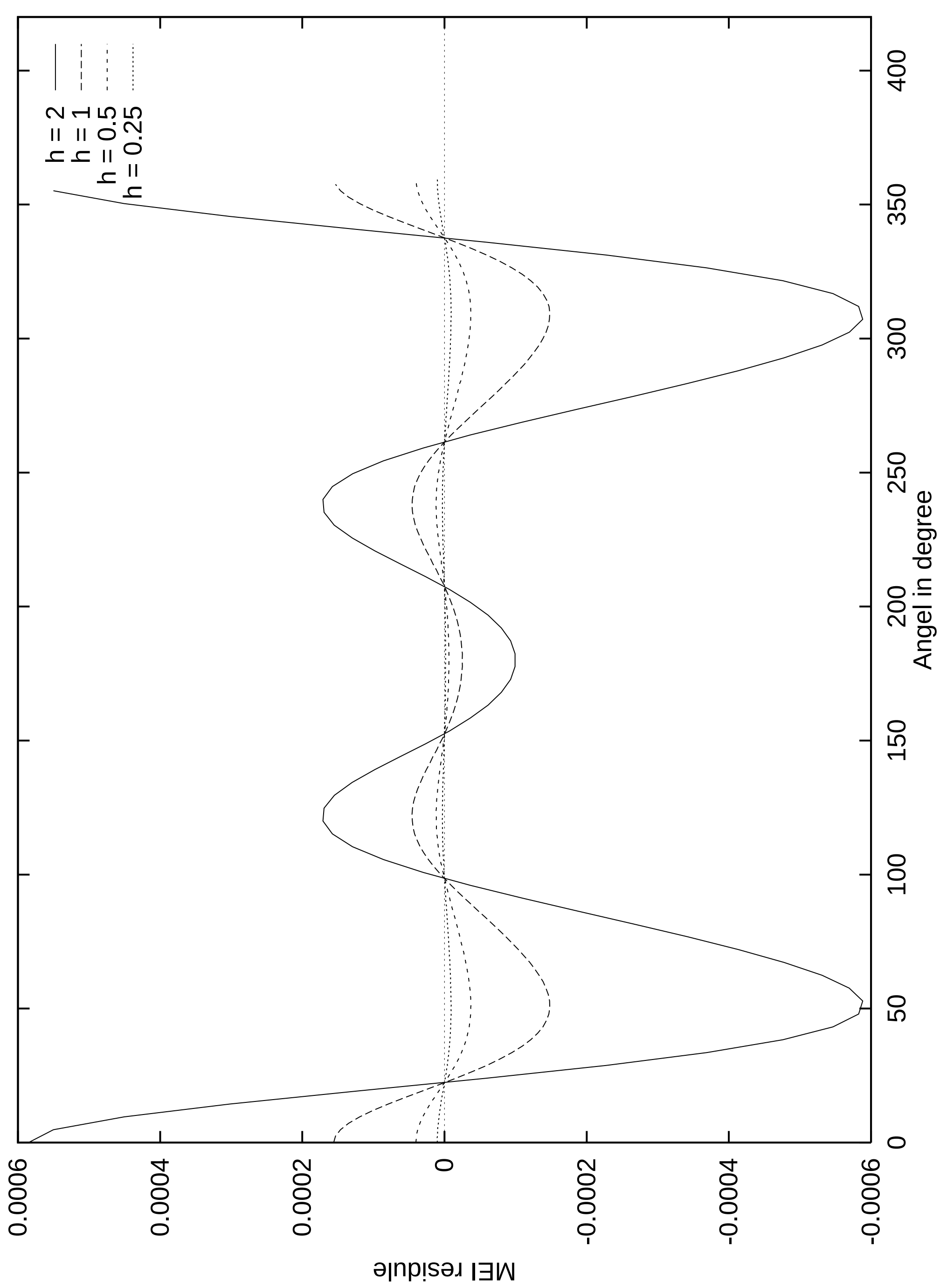

Figure 5: Metron set B: the equal distance point metrons located at the circle with radius $a = 20$

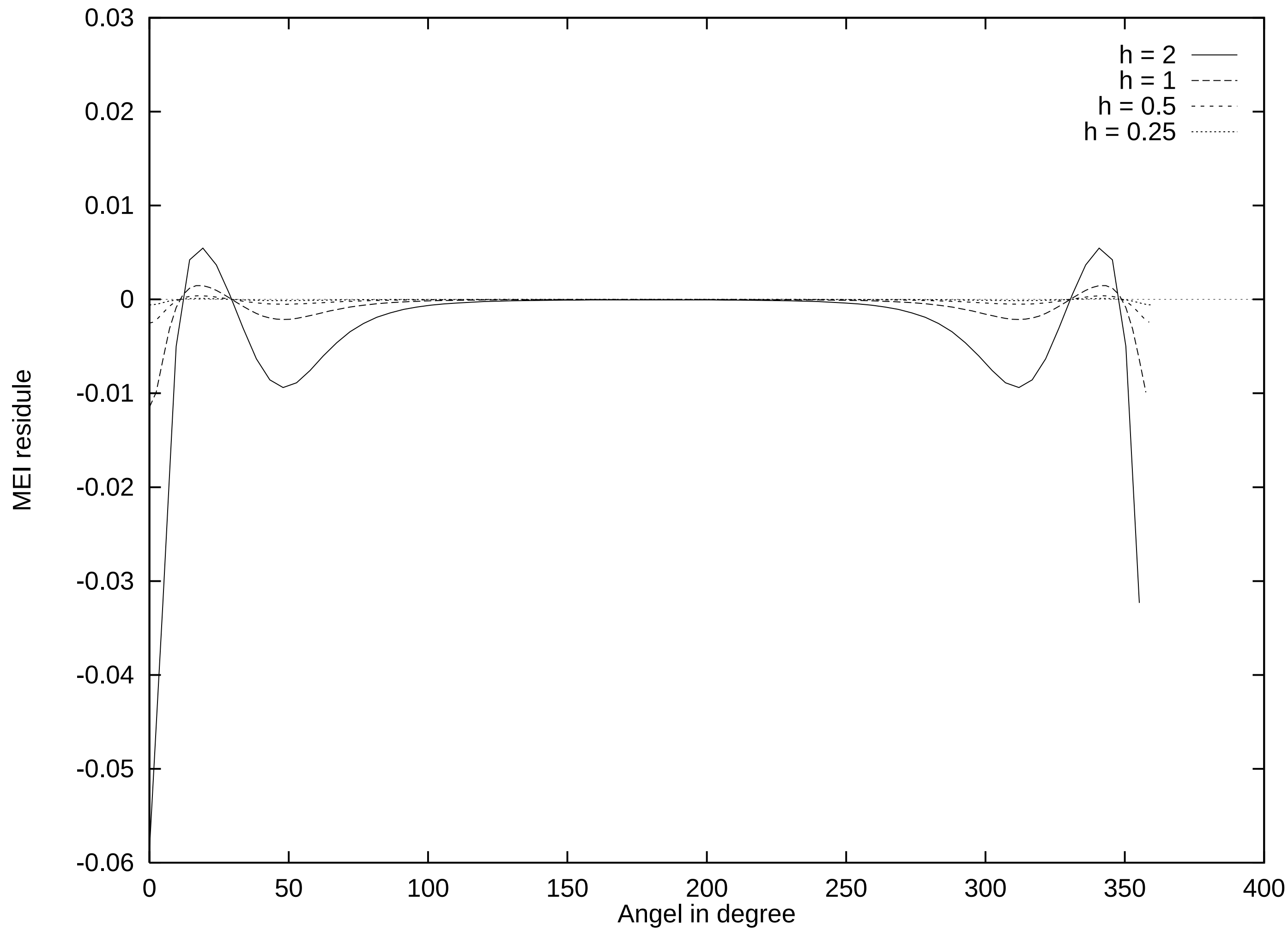

Figure 6: Metron set C: three point metrons in a line along y-axis

Theorem 2. *The consistent condition of the MEI is*

$$||L\phi - M\phi|| = \varepsilon h^2 \tag{24}$$

and the residue ε of the consistent condition is the perturbation of a functional $J(s)$ if MEI coefficients are obtained by using metrons set on the surface of objects

It can be easily derived from Eqs.(6)(16) that

$$\frac{||L\phi - M\phi||}{h^2} = \varepsilon \tag{25}$$

and

$$\varepsilon = \frac{1}{2}\frac{\begin{vmatrix} \phi'_t(\bar r_0) & \phi'_n(\bar r_0) & \phi''_t(\bar r_0) & \phi(\bar r_0) \\ \phi'_{t1}(\bar r_0) & \phi'_{n1}(\bar r_0) & \phi''_{t1}(\bar r_0) & \phi_1(\bar r_0) \\ \phi'_{t2}(\bar r_0) & \phi'_{n2}(\bar r_0) & \phi''_{t2}(\bar r_0) & \phi_2(\bar r_0) \\ \phi'_{t3}(\bar r_0) & \phi'_{n3}(\bar r_0) & \phi''_{t3}(\bar r_0) & \phi_3(\bar r_0) \end{vmatrix}}{\begin{vmatrix} \phi'_{t1}(\bar r_0) & \phi'_{n1}(\bar r_0) & \phi_1(\bar r_0) \\ \phi'_{t2}(\bar r_0) & \phi'_{n2}(\bar r_0) & \phi_2(\bar r_0) \\ \phi'_{t3}(\bar r_0) & \phi'_{n3}(\bar r_0) & \phi_3(\bar r_0) \end{vmatrix}} = \xi \begin{vmatrix} \phi'_t(\bar r_0) & \phi'_n(\bar r_0) & \phi''_t(\bar r_0) & \phi(\bar r_0) \\ \phi'_{t1}(\bar r_0) & \phi'_{n1}(\bar r_0) & \phi''_{t1}(\bar r_0) & \phi_1(\bar r_0) \\ \phi'_{t2}(\bar r_0) & \phi'_{n2}(\bar r_0) & \phi''_{t2}(\bar r_0) & \phi_2(\bar r_0) \\ \phi'_{t3}(\bar r_0) & \phi'_{n3}(\bar r_0) & \phi''_{t3}(\bar r_0) & \phi_3(\bar r_0) \end{vmatrix} \tag{26}$$

where ξ is independent of ϕ. It is known that the real potential ϕ may be expressed as $\phi(r) = \int_\Gamma s(\bar r')G(\bar r, \bar r')d\Gamma'$, where s stands for the source distribution on the surface of the cylinder. Then the residue ε may be expressed as

$$\begin{aligned} \varepsilon &= \xi \begin{vmatrix} \int_\Gamma s(\bar r')G'_t(\bar r_0)d\Gamma' & \int_\Gamma s(\bar r')G'_n(\bar r_0)d\Gamma' & \int_\Gamma s(\bar r')G''_t(\bar r_0)d\Gamma' & \int_\Gamma s(\bar r')G(\bar r_0)d\Gamma' \\ \phi'_{t1}(\bar r_0) & \phi'_{n1}(\bar r_0) & \phi''_{t1}(\bar r_0) & \phi_1(\bar r_0) \\ \phi'_{t2}(\bar r_0) & \phi'_{n2}(\bar r_0) & \phi''_{t2}(\bar r_0) & \phi_2(\bar r_0) \\ \phi'_{t3}(\bar r_0) & \phi'_{n3}(\bar r_0) & \phi''_{t3}(\bar r_0) & \phi_3(\bar r_0) \end{vmatrix} \\ &= \xi \int_\Gamma s(\bar r') \begin{vmatrix} G'_t(\bar r_0) & G'_n(\bar r_0) & G''_t(\bar r_0) & G(\bar r_0) \\ \phi'_{t1}(\bar r_0) & \phi'_{n1}(\bar r_0) & \phi''_{t1}(\bar r_0) & \phi_1(\bar r_0) \\ \phi'_{t2}(\bar r_0) & \phi'_{n2}(\bar r_0) & \phi''_{t2}(\bar r_0) & \phi_2(\bar r_0) \\ \phi'_{t3}(\bar r_0) & \phi'_{n3}(\bar r_0) & \phi''_{t3}(\bar r_0) & \phi_3(\bar r_0) \end{vmatrix} d\Gamma' \end{aligned} \tag{27}$$

It may also be expressed as a functional

$$\varepsilon = J(s) = \int_\Gamma s(\bar r')F(\bar r, \bar r')d\Gamma' \tag{28}$$

So, generally the residue of the consistent condition of the MEI can be expressed as a functional of the source distribution on the surface of the cylinder. It will be shown that the residue will be greatly reduced if the MEI coefficients are determined from the metrons on the surface of the cylinder.

let $T = span\{m_1(\bar{r}'), m_2(\bar{r}'), m_3(\bar{r}')\}$ be the subspace spanned by three metrons $m_1(\bar{r}'), m_2(\bar{r}')$ and $m_3(\bar{r}')$ defined on the surface of the cylinder, then the potentials at the MEI nodes are expressed as

$$\phi_i(\bar{r}_j) = \int_\Gamma m_i(\bar{r}')G(\bar{r}_j, \bar{r}')d\Gamma' \tag{29}$$

where the subscripts i and j stand for the ith metron and the jth MEI node respectively.

Define $\mathbf{P}s$ as the projection of the real source $s(\bar{r}')$ to the metron subspace T, then $s(\bar{r}')$ can be expressed as

$$s = \mathbf{P}s + \delta s \tag{30}$$

where δs is the distance (difference) between s and the subspace T.

Substituting the Eqs.(29)(30) into Eq.(28) 0r (27) and by using the properties of determinant, it is easy to prove that

$$\varepsilon = J(s) = J(\mathbf{P}s + \delta s) = J(\mathbf{P}s) + J(\delta s) = J(\delta s) = \delta J \tag{31}$$

Since the main part of the residue ε being equal to zero, i.e., $J(\mathbf{P}s) = 0$, the residue is greatly reduced. If the metrons are not chosen as the distributions on the surface of the cylinder, the conclusion Eq.(31) is not valid. It will make serious error in the solution.

If the real source s is just a linear combination of the three metrons, in other words, $s \in T$, it is obvious that $\delta s = 0$ and then $\varepsilon = 0$. This means the MEI rigorously simulated the truncated boundary condition without any error. In fact, generally the real source $s \notin T$, and there must be a small residue $\varepsilon = \delta J$. The residue decreases with the decrease of the difference δs between the real source and the metron subspace T. Therefore, we should choose the metrons taking some physical concept into consideration, so as to minimize the residue. Fortunately, the first several terms of the sinusoidal sequence are suitable metrons for most problems.

It should be noticed that only three metrons are considered in the discussion above. If the metron subspace is spanned by more metrons, i.e., $T = span\{m_1, m_2, ..., m_k\}$, the difference δs and then the residue δJ will be further decreased. For this situation, the MEI coefficients will be obtained by a least square procedure. As discussed in [HML94], piecewise functions are also a good choice of metrons for most problems, especially for three dimensional problems where continuous functions are hard to be defined.

We also use the previous example to demonstrate this theorem. Note that in the above example, we have three sets of metrons, with only one of them (set A) on the surface of the cylinder. We can see from Fig 4 to 6 that the residue of set A is much smaller than that of set B, and is two orders of magnitude smaller than that of set C. The residue of set B is smaller than set C because metrons in set B can approximate the shape of surface better than set C. This verifies theorem 2.

Theorem 3. *If the PDE $L\phi = 0$ satisfies the condition of the maximum theorem, and the FD equations with the error bounded by $O(h^4)$ are applied to the interior mesh nodes, MEIs to the truncated boundary nodes, then $||\phi - \phi'|| \leq \xi_1||\delta J|| + \xi_2 h^2$, where ϕ and ϕ' are the* *accurate solution and the approximate solution of MEI respectively,* ξ_1 *and* ξ_2 *are independent of* h.

It is well known that the consistent condition for the FD equation with error bounded by $O(h^4)$ is $||L\phi - D\phi|| = O(h^4)$, where D is the finite difference operator. And the consistent condition for MEI has been shown in theorem 2. It is straightforward to get the conclusion

$$||\phi - \phi'|| \leq \xi_1 ||\delta J|| + \xi_2 h^2 \tag{32}$$

according to maximum principle [Smi85, BL84].

This theorem states that the accuracy of the solution of MEI does not increase with the decreasing of step size h, however, if metrons are chosen properly on the surface of object, the MEI still ensures the accuracy of the solutions. According to Lax theorem [Smi85, BL84], which essentially states the convergence property is equivalent to stability property given that the consistent condition is satisfied. Therefore, this theorem somehow described the convergence property of MEI.

Fig. 7 shows the relative error of potential distribution between accurate solution and MEI solutions with different step sizes. It can be seen that the error does not decrease when step size decreases from 2 to 0.25. However, when the step size is 2 which means only two layers of mesh are used outside the cylinder, the solution is already accurate enough with 0.2% relative error. On the other hand, if metrons are chosen improperly such as the metron set B and C as mentioned above, the relative error will be orders of magnitude larger than that by using the metron set A, as shown in Fig. 8, which also verifies conclusion of theorem 2. Furthermore, improperly chosen metrons (such as metron set C) will seriously degenerate the solution as shown in Fig. 9.

4 Conclusion

In this paper, we proved three theorems and got some important conclusions which gave deep insight into Measured Equation of Invariance. The first theorem stated that the residue of MEI equation is independent of excitation with error bounded by $O(h^2)$ where h is the discretization step size. The second theorem stated that if metrons are set on the surface of object, the residue of the consistent condition of MEI is proportional to the perturbation of a functional with respect to the source distribution on the surface of object. The third theorem stated that the error of the MEI solution does not decrease with the decreasing of step size h, however, if metrons are chosen properly on the surface of object, the MEI still ensures the accuracy of the solutions. Numerical experiments verified the three theorems and showed that if properly metrons are chosen on the surface of object, MEI will achieve enough accurate solution even with only two layers of mesh outside the object; on the other hand, improperly chosen metrons will seriously degenerate the final solution. Besides, the smaller the distance between the accurate source distribution and the subspace spanned by the metrons, the better the solution, therefore, we should choose metrons with some *prior* knowledge.

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