Simultaneous Construction of Refutations and Models for Propositional Formulas

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Abstract
Methodology is developed to attempt to construct simultaneously either a refutation or a model for a propositional formula in conjunctive normal form. The method exploits the concept of “autarky”, which was introduced by Monien and Speckenmeyer. Informally, an autarky is a “self-sufficient” model for some clauses, but which does not affect the remaining clauses of the formula. Whereas their work was oriented toward finding a model, our method has as its primary goal to find a refutation in the style of model elimination. It also finds a model if it fails to find a refutation, essentially by combining autarkies. However, the autarky-related processing is integrated with the refutation search, and can greatly improve the efficiency of that search even when a refutation does exist. Unlike the pruning strategies of most refinements of resolution, autarky-related pruning does not prune any successful refutation; it only prunes attempts that ultimately will be unsuccessful; consequently, it will not force the underlying search to find an unnecessarily long refutation. A game characterization of refutation search is introduced, which demonstrates some symmetries in the search for a refutation and the search for a model. Limited experience with a prototype implementation is reported, and indicates the possibility of developing high-performance refutation methods that are competitive with recently reported model-searching methods. Considerations for first-order refutation methods are discussed briefly.

Key Words
Satisfiability, Boolean formula, propositional formula, autarky, resolution, refutation, model, theorem proving, model elimination.

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1 Introduction

The decision problem of Boolean, or propositional, satisfiability is the “original” NP-hard problem. We assume the reader is generally familiar with it. We shall consider exclusively propositional formulas in conjunctive normal form (CNF), also called clause form. Each clause is a disjunction of literals, and clauses are joined conjunctively. A closely related problem is to determine validity of a formula in disjunctive normal form. As language recognition problems, satisfiability is in NP, while validity is in co-NP. However, as decision problems, they are essentially equivalent, as both “yes” and “no” answers must be produced. As a practical consideration, a program may be required to produce “evidence”, or a “certificate”, to back up its decision. If critical decisions will be based on the program’s output, such evidence should permit an independent, and straightforward, verification of the program’s conclusion. To our knowledge, no previously existing implementations can produce evidence for both “yes” and “no” decisions.

Two basic methods have been developed for satisfiability testing: refutation search and model search.

1. Refutation search seeks to discover a proof that a formula is unsatisfiable, usually employing resolution. If a complete search for a refutation fails, the formula is pronounced satisfiable. Model elimination and SL-resolution typify these methods.

2. Model search seeks to discover a satisfying assignment, or model, for the formula. If a complete search for a model fails, the formula is pronounced unsatisfiable. The DPLL algorithm, due to Davis, Putnam, Loveland and Logemann [DP60, DLL62] is the basis for many modern refinements. A different approach is to treat the problem in terms of integer linear programming [BJL86, JW90, HF90, HHT94].

Several methods use incomplete model searches, so they can only report “don’t know” and give up based on resource limits, when they fail to discover a model [SLM92, GW93, SKC95]. However, they have succeeded in finding models on much larger formulas than current complete methods can handle.

Current propositional methods are “one-sided” in the sense that they can produce (“evidence”, or a “certificate”, for one side of the decision problem, but not the other. Refutation methods do not produce a model on satisfiable formulas, and model-search methods do not produce a refutation on unsatisfiable formulas. This paper introduces an integrated approach that simultaneously searches for either a refutation or a model.

In the first-order arena there has been some work on searching for a model as well as a refutation, but it does not seem to carry over effectively to the propositional case. The method of Fermüller and Leitsch first performs hyper-resolution to saturation, then if the empty clause has not been derived, there must be a model [FL93]. Their main focus is theoretical, to show that certain classes of first order formulas are decidable, and they do not investigate a specific method to extract a model. This strategy is not practical in the propositional case. The method of Caferra adds equations soundly to the original formula and shows that for some classes the refutation search is guaranteed to terminate [Caf93]. Since the equations are based on unifying substitutions, there is no apparent way to use this method in the propositional case. Finally, failure caching on first-order Horn formulas has been reported [Elk89, AS92], but propositional Horn formulas are not challenging.

Several high performance satisfiability testers have been reported in recent years [Lar92, CA93, ZS94, SFS95, inter alia]. See also papers and bibliographies from the DIMACS Second Implementation Challenge [DABC95, JSD95, Pre95, VGT96]. Interestingly, they have primarily, if not exclusively, been based upon the model-search paradigm. This contrasts with the situation of first-order theorem provers, which are primarily refutation-based.

One problem with existing propositional refutation methods is illustrated dramatically in Figure 1. Model elimination (regarded as one of the most efficient refutation strategies) is able to solve unsatisfiable random
Figure 1: Comparative performances of model elimination on unsatisfiable and satisfiable random 3CNF formulas. "??" indicates that none of these formulas were solved with available resources. On each formula one refutation was attempted with the first clause as top clause. Implementation and experimental details are given in later sections, along with results for the new Modoc algorithm.

3CNF formulas with up to 100 variables, but it bogs down on satisfiable formulas about at 15 variables. It should be noted that, for modern model-search methods, these formulas (both unsatisfiable and satisfiable) are considered easy at 100 variables and trivial at 20 variables. Three reasons have been mentioned for the lack of high performance propositional refutation systems:

1. The search space is too large, particularly for satisfiable formulas, as shown in Figure 1.

2. The method cannot produce models.

3. The most efficient methods are only guaranteed to find a refutation when some top clause is known to be in a minimal unsatisfiable subset of clauses; in some applications such a top clause is not known a priori.

The integrated approach of this paper addresses all three of these problems.

1.1 Summary of Results

A game characterization of refutation search is introduced (Section 4.2). This game, called the PDST game, demonstrates some symmetries in the search for a refutation and the search for a model. The game tree can also be viewed as a certain and-or tree whose evaluation indicates whether a refutation exists.

Investigation of PDST game strategies leads to properties of the game that involve autarkies (Section 5), which are the main results of the paper. A method for recursively constructing autarkies is developed. It is shown that clauses satisfied by such autarkies can be pruned from the refutation search. The noteworthy result here is that clauses pruned by such autarkies cannot participate in any successful refutation. This contrasts with the weaker property of most resolution refinements that some successful refutation remains after pruning, but the remaining refutation may be significantly longer than some that were pruned [LMG94]. Section 6 describes an algorithm, called Modoc, that essentially combines weak model elimination with autarky construction and autarky-based pruning.
Figure 2: Model elimination search for Example 1.1. (Left) Search fails at lowest \( \neg c \) goal. (Right) After backtracking to alternative choices at \( \neg a \) goal.

Model elimination provides an optional lemma creation mechanism. Section 7 briefly reviews previous work, and discusses the compatibility of autarky pruning and lemma creation. It also introduces and describes a strategy for "quasi-persistent" lemmas.

A prototype implementation has been programmed in Prolog. Limited experimental results are reported in Section 8. They indicate the possible feasibility of building a high-performance resolution-based solver for propositional CNF formulas, but considerable work remains to be done. Relationships to first-order theorem proving are mentioned briefly at the end of Section 4.1 and in Section 5.2. Section 9 draws conclusions and indicates future directions.

For readers familiar with model elimination adapted to tree structures, we now give a motivating example of how autarky pruning works. Other readers may wish to come back to this example after reading more of the expository material in Section 3.

**Example 1.1:** The formula \( S \) consists of all 3CNF clauses on variables \( a, b, \) and \( c \), except for the all positive clause, as shown at the top of Figure 2. Suppose the top clause is all negative. Let us trace out a model elimination search for a refutation. As shown on the left of Figure 2, literal \( \neg a \) resolves with clause \([a, \neg b, \neg c]\), then literal \( \neg b \) resolves with clause \([a, b, \neg c]\). In the latter clause, literal \( a \) can be reduced with the ancestor (or A-literal) \( \neg a \), as indicated by the boxed "A". Literal \( \neg c \) remains to be refuted.

At this point the search procedure is stuck, because each clause containing literal \( c \) also contains an ancestor literal, so it is not eligible for extension at this point in the tree. This rule is variously called "preadmissibility," "tightness," and "regularity."

If we stop and reflect on the meaning of this failure, it can be stated in words, as follows: every clause containing the literal \( c \) also contains some other literal that is an ancestor (or A-literal) on this path in the tree being constructed. Therefore, every clause containing the literal \( c \) is satisfied by a partial assignment consisting of the ancestors (A-literals) on this branch, specifically:

\[
M = \{ \neg a, \neg b, \neg c \}.
\]
(In this case the partial assignment happens to be a total assignment.) But obviously, every clause containing the literal $-c$ is also satisfied by $M$, so we conclude that every clause involving the variable $c$ is satisfied by $M$.

Model elimination now backtracks and looks for another clause that resolves with $-b$, and does not contain the ancestor $-a$. There are none. We can now extend the conclusion of the previous paragraph to say that every clause containing either of the variables $b$ or $c$, either positively or negatively, is satisfied by the partial assignment $M$.

Model elimination again backtracks and looks for another clause that resolves with $b$, and does not contain the ancestor $-a$. There are none. We can now extend the conclusion of the previous paragraph to say that every clause containing either of the variables $b$ or $c$, either positively or negatively, is satisfied by the partial assignment $M$.

Model elimination again backtracks and looks for another clause that resolves with $a$ (Figure 2, right). There are two such clauses, as indicated. The standard algorithm continues trying to construct a refutation using one of these clauses, and if that fails, it tries the other. But notice that both of these clauses are satisfied by the partial assignment $M$ mentioned above.

After a few moments thought, we can predict that these refutation attempts must fail, without carrying out the search. Intuitively, the reason is that we cannot use a clause that is satisfied by $M$ to "get outside of $M". For example, if $-a$ is extended with clause $[a, b, -c]$, then (possibly among others) there is a goal whose literal is in $M$, in this case, $-c$. This goal must be refuted by extension. The (hypothetical) extension clause contains $c$, and every clause containing $c$ is known to be satisfied by $M$, so some other literal of that hypothetical clause is in $M$. In this case, the only other possibility is $-b$, because $-a$ and $-c$ are ancestors at this point. Now $-b$ must again be refuted by extension: reduction is impossible because the ancestors on this branch are all in $M$. But again, the (second hypothetical) extension clause must be satisfied by $M$, so it must generate another goal that is in $M$, etc., until some goal is generated that has no eligible extensions.

Finally, we conclude that the partial assignment $M$ satisfies all clauses in which any of the variables $a$, $b$, or $c$ appears. This conclusion holds up even if we add additional clauses to $S$ that do not involve the variables $a$, $b$ and $c$. We call such a partial assignment an autarky (see Definition 5.1).

This example illustrates, in an over-simplified way, the two main themes of the paper:

1. Autarky analysis can predict that certain refutation attempts must fail;

2. A model for a satisfiable formula can be constructed as a series of autarkies.

2 Preliminaries

Standard terminology for conjunctive normal form (CNF) formulas is used. A finite set of propositional variables is fixed throughout the discussion. The term "propositional variable" is abbreviated to "variable" when no confusion can result.

Definition 2.1: (literal, clause, formula) A literal is a positive variable $x$, or a negated variable $-x$. Literals $x$ and $-x$ are complementary. The complement of literal $q$ is denoted $\neg q$, whether $q$ is positive or negative; i.e., double negations are simplified away.

A clause is a disjunction of zero or more literals, represented simply as a set of literals. Of special interest are the empty clause, denoted by $\emptyset$, representing false, and unit clauses, consisting of exactly one literal. A clause consisting of literals $p_1, \ldots, p_k$ ($k \geq 1$) is denoted as $[p_1, \ldots, p_k]$.

A CNF formula (formula for short, since only CNF formulas are considered) is a conjunction of zero or more clauses, represented simply as a set of clauses (or a multiset, if duplicate clauses occur). The empty formula represents true. If $F$ is a formula, then $\text{lit}(F)$ denotes the set of all literals composed from propositional variables occurring in $F$. \(\square\)
**Definition 2.2: (assignment, satisfaction, model)** A partial assignment is a partial function from the set of variables into \{false, true\}. This partial function is extended to literals, clauses, and formulas in the standard way. If the partial assignment is a total function, it is called a total assignment, or simply an assignment.

A clause or formula is satisfied by a partial assignment if it is mapped to true; a formula is satisfiable if it is satisfied by some partial assignment; otherwise, it is unsatisfiable. A partial assignment that satisfies a formula is called a model of that formula. (Thus any model can be extended to a total assignment that is also a model by assigning arbitrary values to the unassigned variables.)

“Consistent” is a synonym for “satisfiable”, and “inconsistent” is a synonym for “unsatisfiable”. This paper will only apply the terms “consistent” and “inconsistent” to formulas consisting entirely of unit clauses and/or the empty clause.

A partial assignment is conventionally represented by the (necessarily consistent) set of unit clauses that are mapped into true by the partial assignment. Note that this representation is a very simple formula. Set-forming braces are omitted sometimes to streamline notation.

**Definition 2.3: (overloading, disjoint union, subset difference)** If \( q \) denotes a literal, then in a setting where a clause is required \([q]\) may be written simply as \( q \). Similarly, if \( C \) denotes a clause, then in a setting where a formula is required \( f \) may be written simply as \( C \).

In a setting where a partial assignment is required, \( \{q_1, \ldots, q_k\} \) denotes the formula of unit clauses \([q_1], \ldots, [q_k]\).

Disjoint union is denoted by “+”. This partial binary function is defined precisely when its operands are disjoint sets, and produces their union.

Subset difference is denoted by “−”. This partial binary function is defined precisely when its second operand is a subset of its first operand, and produces the set difference.

As an example of overloading, if \( F \) denotes a formula (which does not contain the unit clause \([q]\)), then \( F + q \) denotes \( F \cup \{[q]\} \).

The limited definitions of “+” and “−” on sets permits these symbols to be used more algebraically. The following identities are easily established:

**Lemma 2.1:** When the sets \( A \) through \( D \) are such that the indicated partial functions are defined,

\[(A - B) + B = A \quad (C + D) - D = C\]

**Definition 2.4: (strengthened formula)** Let \( M \) be a partial assignment for formula \( S \). The clause \( C|M \), read “\( C \) strengthened by \( M \)”, is the (possibly empty) set of literals

\[C|M = C - \{q \mid q \in C \text{ and } \neg q \in M\}\]

The formula \( S|M \), read “\( S \) strengthened by \( M \)”, is the (possibly empty) set of clauses

\[S|M = \{C|M \mid C \in S \text{ and } C \text{ contains no literal of } M\}\]

**Example 2.1:** Let \( S \) consist of \([a, b], [\neg a, c], \text{ and } [b, d] \). Then \( S|a = \{[c], [b, d]\} \), and \( S|\{a, c\} = \{[b, d]\} \).
The next equivalence is a special case of Shannon factorization that has been exploited in many works [DP60, DLL62, AB70].

**Lemma 2.2:** Formula $S$ is logically equivalent to $(x \land S \setminus \{x\}) \lor (\neg x \land S \setminus \{\neg x\})$. □

**Definition 2.5:** (formula size) For purposes of induction proofs, the size of a formula is usually defined to be the number of occurrences of literals in the formula, and is denoted as $|S|$.

$$|S| = \sum_{C \in S} |C|$$

where $|C|$ is the usual set cardinality. □

### 3 Clause and Derivation Trees

Trees are now recognized as the most appropriate data structures for representation of linear resolution derivations. This section describes the tree data structures we shall use, which are chosen especially for propositional resolution.

Two refinements of resolution were proposed independently, called model elimination [Lov69], and SL-resolution [KK71], which are closely related [Lov72]. The model elimination data structure was described as a “chain”. Within the chain could be found the most recently derived clause, as well as other information about the proof. Minker and Zanon appear to be the first of several researchers to recognize that the tree data structure was more appropriate than a chain [MZ82]. Recently, Letz et al. have given a unified view of the methods of tableau calculus and clause trees [LMG94]. These proposals were oriented toward first-order applications. Our trees differ somewhat because propositional resolution does not need to take into account substitutions, which fact permits simplifications in both refutations and searches for refutations. The following technical definitions are illustrated in Example 3.1 and Figure 3.

**Definition 3.1:** (clause-goal tree, goal ancestor) Let a set $S$ of propositional clauses be given (i.e., a formula). Let $\top$, called *verum*, be a symbol distinct from all propositional variables.

A clause-goal tree is a bipartite directed tree with two classes of nodes, called *clause* nodes and *goal* nodes. That is, a clause node may have only goal nodes as children and vice versa. Edges are directed from the root to the leaves. Recall that a branch of a tree is a path from the root to a leaf. The tree is unordered in the sense that the order of any node’s children is immaterial.

Each clause node is labeled with a clause of $S$, and each goal node is labeled with a literal of $lits(S)$, or with $\top$. Usually, a node is identified with its label, but when it is necessary to name a specific node, we assume some structural naming scheme. We write $v(q)$ to denote the goal node whose structural name is $v$ and whose label is $q$, and write $w[C]$ to denote the clause node whose structural name is $w$ and whose label is $C$.

A goal ancestor of a node $v$ is a goal node on the path from the root to $v$, including $v$ itself if it is a goal node. Since clause ancestors are not significant, we shall refer to goal ancestors simply as ancestors. The set of all goal ancestors of $v$ is denoted as $\text{ancs}(v)$. While $\text{ancs}(v)$ is technically a set of nodes, it can also be considered as a set of unit clauses made from the nodes’ literal labels, i.e., a formula. □

**Definition 3.2:** (propositional derivation tree (PDT), PDT extension) Let $S$ and $\top$ be as in Definition 3.1. Throughout this definition all literals are assumed to be in $lits(S)$ and all clauses are assumed to be in $S$.

A propositional derivation tree (PDT) is a clause-goal tree that can be constructed according to the following inductive definition:
Figure 3: Propositional derivation trees (PDTs) discussed in Example 3.1. Left: top-clause PDT for $[b, \neg f]$. Center: single-clause PDT with root $b$ and clause child $[\neg b, \neg d, f]$. Right: their combination.

1. The tree consisting only of a goal node, labeled with either a literal or $\top$, is a PDT.

2. A tree with the following structure is a PDT, and is called a single-clause PDT.
   
   (a) The root is a goal node, labeled with $q$, and has a single child labeled with clause $C$.
   
   (b) $C = [\neg q, p_1, \ldots, p_k]$, where $k$ may be zero (in case $C$ is a unit clause), and each literal $p_i$ is different from $q$.

   (c) Node $C$ has $k$ children, labeled $p_i$, for $1 \leq i \leq k$. The children of $C$ are leaves.

3. A tree is a PDT, and is called a top-clause PDT for $C$, if it differs from a single-clause PDT (above) in that the root is labeled with $\top$ and there is a leaf corresponding to every literal of $C$. In this case $C$ is called the top clause of the PDT. (Note that this fits the format of a single-clause PDT if we imagine that $\top$ is another variable and $C$ contains an “invisible” literal $\neg \top$.)

4. If $T_1$ is a PDT with a leaf $v(q)$, and $T_2$ is a PDT with root $q$, then a new PDT $T_3$ may be formed as follows. Let $T'_2$ be obtained from $T_2$ by pruning all subtrees that are:

   (a) rooted at a goal node that is complementary to a unit clause of $\text{ance}(v)$; or
   
   (b) rooted at a clause node that contains a literal in $\text{ance}(v)$.

   Then $T_3$ is obtained by replacing node $v(q)$ in $T_1$ by $T'_2$. If $T_2$ is a single-clause PDT with clause $C$, then the operation that constructs $T_3$ is called a PDT extension of $T_1$ at $v(q)$ by $C$.

From the definition we see that every PDT is rooted with a goal node, and every goal node either is a leaf or has exactly one child. Also, it is easy to see that a clause node $u[C]$ is a leaf in a PDT if and only if every literal in $C$ is complemented in $\text{ance}(u)$.

**Definition 3.3: (refutation)** If $v(q)$ is the root of a PDT that contains only clause nodes as leaves, then this PDT is called a refutation of $q$ with respect to $S$. If $q = \top$, it is called simply a refutation of $S$. (These terms are justified in Theorem 3.2.)
Example 3.1: Figure 3 shows PDT examples. The PDT at the right is obtained by PDT extension using the other two. The left side of Figure 2 shows a clause node in which implicit reduction has occurred, as indicated by the boxed “A”. (This box is a notation, but not a structural part of the tree.)

By construction, for any node $v$ of a PDT, $\text{ancs}(v)$ (excluding $T$) is a consistent set of unit clauses, i.e., a partial assignment. Consequently, if $V$ denotes the set of propositional variables, then the PDT has depth at most $2|V| + 1$ (counting both clause and goal nodes, and defining the depth of a tree with one node to be zero).

Also by construction, if $w[C]$ is a clause node of a PDT, then the literals of $C$ are disjoint from those in $\text{ancs}(w[C])$, and the variables occurring in any children of $C$ are disjoint from the variables occurring in $\text{ancs}(w[C])$. However, $C$ may contain the complement of a literal in $\text{ancs}(w[C])$.

Let us point out that the terminology “PDT extension” as defined above is a combination of model elimination operations called “extension” and “reduction”. In some contexts “reduction” has been called “ancestor resolution”, “subsumption resolution”, or “s-resolution”. In the propositional framework, model elimination’s “extension” operation need not be considered where a “reduction” is possible, so it is useful to combine “reductions” into the “PDT extension”. In a first-order framework, both “extension” and “reduction” normally must be considered, due to differing unifiers, so the goal must be created. A PDT extension can be viewed as a model elimination extension followed by as many reductions as possible on the literals of $C$. To make a PDT tree correspond more closely to a model-elimination tree [MZ82] or a connection tableau [LMG94], one can add a third node type, “reduction”, that is automatically attached as the only child of any goal node for which reduction is applicable.

Lemma 3.1: Every PDT can be built from a single-clause tree or a top-clause tree by a series of PDT extensions.

Proof: Note that each PDT extension adds exactly one clause node to the tree. A straightforward induction shows that $T_3$ in Definition 3.2 can be created by concatenating the series of PDT extensions for $T_1$ and $T_2$, eliminating those in $T_2$ whose clause nodes were pruned from $T_1$. Each PDT extension that originated in $T_2$ (at node $v$ using clause $C$) and remains in $T_3$ is “legal” in $T_3$ because the ancestors of its clause node in $T_3$ are the union of those in $T_2$ and $\text{ancs}(v)$ in $T_1$. Therefore, no literal in $C$ occurs in its $T_3$ ancestors, and the children of this clause node are exactly those literals in $C$ that are not complemented by the $T_3$ ancestors.

For purposes of intuition, a “goal node” means that the goal of the derivation is to refute the literal in the node, not to validate it. The term “refutation” in Definition 3.2 is justified by the following theorem, which essentially states that propositional weak model elimination is sound. Although this is already well known [Lov69, MZ82, LMG94], we outline the proof in the interest of self-containment, and to validate our new formulation, given in Definitions 3.2 and 3.3.

Theorem 3.2:

(A) If there is a PDT $T$ that is a refutation of $q$ w.r.t $S$, then $S$ has no model in which $q$ is true.

(B) If there is a PDT that is a refutation of $S$, then $S$ is unsatisfiable.

Proof: The proof of (A) is by induction on the $||S||$ (Definition 2.5). The base case, $||S|| = 1$, is immediate, as is any case in which the child of the root is the unit clause $\neg q$. Otherwise, let $C = [\neg q, p_1, \ldots, p_k]$, where $k \geq 1$, be the clause child of the root. Form $S'$ by adding the unit clause $q$ to $S$ and removing $C$. Then
This section describes propositional derivation search trees. These structures have some similarity in purpose to tableau search trees [LMG94]. However, there are some essential structural differences, as discussed at the end of Section 4.1, which are again based on exploiting the simplifications available in

\[
S = \begin{array}{c}
b, \neg f \\
\neg b, \neg d, f \\
c, d \\
c, \neg d \\
c, \neg f \\
\neg c, e \\
\neg c, \neg c \\
\end{array}
\]

\[
\begin{array}{c}
\top \\
b, \neg f \\
\neg f \\
\neg b, \neg d, f \\
\neg d \\
f, \\
\neg b, \neg d, f \\
\neg b \\
\neg d \\
c, d \\
c, \neg f \\
\neg c, e \\
\neg c, \neg c \\
\end{array}
\]

Figure 4: A completed propositional derivation search tree (PDST) with top clause \([b, \neg f]\).

\[
||S'|| < ||S||, \text{ so the inductive hypothesis applies. Denote by } T_q \text{ the single-clause tree with root } \neg q \text{ and unit clause } [q]. \text{ By Definition 3.2, the clause node has no children.}
\]

For \(1 \leq i \leq k\), let \(T_i\) be the subtree of \(T\) rooted at the goal node \(p_i\) that is a child of \(C\). Tree \(T_i\) can be transformed into a PDT \(T'_i\) for \(S'\) by attaching a copy of \(T_q\) beneath every clause node whose clause contains \(\neg q\). (If \(D\) is such a clause node, note that \(\text{anc}_D(D)\) in \(T'_i\) is exactly \(\text{anc}_D(D)\) in \(T\) with \(q\) deleted. Therefore, \(D\) in \(T'_i\) requires an extra child \(\neg q\) to satisfy the PDT constraints.) Since no leaf goal nodes are added in this transformation, \(T'_i\) is a refutation of \(p_i\) w.r.t. \(S'\). By the inductive hypothesis, \(S'\) has no model in which \(p_i\) is true.

Clearly, for any \(i\) \((1 \leq i \leq k)\), if \(S\) has a model with \(q\) and \(p_i\) both true, so does \(S'\), and this is impossible due to \(T'_i\) above. It is also impossible that \(S\) has a model with \(q\) true and all \(p_i\) false, for \(1 \leq i \leq k\), due to clause \(C\). So part (A) is proved. Part (B) follows easily from part (A).

4 Search Trees

This section describes "propositional derivation search trees". These structures have some similarity in purpose to tableau search trees [LMG94]. However, there are some essential structural differences, as discussed at the end of Section 4.1, which are again based on exploiting the simplifications available in
the propositional logic. We shall show that a propositional derivation search tree can be regarded as a game
tree, or an and-or tree, whose evaluation determines the satisfiability of the underlying formula.

Basicallly, a propositional derivation search tree is a clause-goal tree that differs from a PDT
(Definition 3.2) only in that a nonleaf goal node, instead of having one clause child, has a clause child for each clause
that might appear in that position in a PDT. It is convenient to visualize a PDST as a three-dimensional
tree, in which goal children of a given clause are arranged left to right, and clause children of a given goal
are arranged front to back (see Figure 4, discussed next).

Example 4.1: Consider the clause set $S$ with top clause $[b, \neg f]$, where

$$S = \{[b, \neg f], [\neg b, \neg d, f], [c, d], [c, \neg d], [c, \neg f], [\neg c, c], [\neg c, \neg c]\}$$

The PDST appears in Figure 4. Observe that $[\neg c, \neg d]$ does not occur in the PDST. The boxed “A”s are not
a structural part of the tree, but are notations to indicate literals that are subject to implicit reduction.

### 4.1 Definitions and Basic Properties

This section states the precise definition and basic structural properties of propositional derivation search
trees, and illustrates them with examples.

**Definition 4.1: (propositional derivation search tree (PDST))** Let $S$ and $T$ be as in Definition 3.1
and Definition 3.2. Throughout this definition all literals are assumed to be in $\text{flts}(S)$ and all clauses are
assumed to be in $S$.

A *propositional derivation search tree* (PDST) is a clause-goal tree that can be constructed according to
the following inductive definition:

1. The tree consisting only of a goal node, labeled with either a literal or $T$, is a PDST.

2. A tree with the following structure is a PDST, and is called a *single-goal PDST for $q$.*

   a. The root is a goal node, labeled with $q$, and has clause children labeled with clauses $C^{[j]}$ for all
      $C^{[j]}$ that meet the next condition.

   b. $C^{[j]} = \{\neg q, p_1^{[j]}, \ldots, p_k^{[j]}\}$, where $k_j$ may be zero (in case $C^{[j]}$ is a unit clause), and each literal
      $p_i^{[j]}$ is different from $q$.

   c. Node $C^{[j]}$ has $k_j$ children, labeled $p_i^{[j]}$, for $1 \leq i \leq k_j$. The children of $C^{[j]}$ are leaves.

3. A tree is a PDST, and is called a *single-clause PDST for $C$*, if it is a top-clause PDT for $C$ (Definition 3.2-3).

4. If $T_1$ is a PDST with a leaf $v(q)$, and $T_2$ is a PDST with root $q$, then a new PDST $T_3$ may be formed
   as follows. Let $T_2'$ be obtained from $T_2$ by pruning all subtrees that are:

   a. rooted at a goal node that is complementary to a unit clause of $\text{ancs}(v)$ (Locations where such
      pruning occurred are marked with a boxed “A” in diagrams.);

   b. rooted at a clause node that contains a literal in $\text{ancs}(v)$.

   Then $T_3$ is obtained by replacing node $v(q)$ in $T_1$ by $T_2'$. If $T_2$ is a single-goal PDST with clause $C$,
   then the operation that constructs $T_3$ is called a *PDST extension of $T_1$ at $v(q)$.*
Figure 5: PDST trees, as discussed in Example 4.2. Left: single-goal PDSTs for $\neg a$ and $\neg b$. Right: their combination.

If the root of a PDST is a literal $q \in \text{lit}(S)$, it is called a PDST for $S$ with top goal $q$; if the root is $\top$ and its child is $w[C]$, the tree is called a PDST for $S$ with top clause node $w[C]$, or with top clause $C$. □

Again, it is easy to see that a clause node $w[C]$ is a leaf in a PDST if and only if every literal in $C$ is complemented in $\text{ancs}(w)$.

Example 4.2: Figure 5 illustrates Part 4 of Definition 4.1. Two PDSTs on the left are combined to produce the one on the right. In the result, subtrees rooted at clause nodes containing $\neg a$ are pruned because $\neg a$ is now an ancestor. In the surviving clause of the lower tree, the subtree rooted at the goal node containing $a$ is pruned for the same reason. The boxed “A” is not a structural part of the tree, but is a notation to indicate a literal that is subject to implicit reduction. □

Definition 4.2: (failed goal node, completed PDST) A failed goal node in a PDST is a leaf node $v(q)$ such that every clause of $S$ that contains $\neg q$ also contains some literal in $\text{ancs}(v(q))$; the branch ending at $v(q)$ is called a failed branch.

A PDST is said to be completed if every leaf that is a goal node is failed. In this case any attempt at a PDST extension results in the entire extension tree being pruned, leaving the original tree. □

Definition 4.3: (universal PDST) For a given formula $S$ a universal PDST for $S$ is constructed as follows: for each clause $C^{(j)} \in S$ create a completed PDST for $S$ with $C^{(j)}$ as top clause, then merge all their roots. □

We now state some structural properties of PDSTs.

Lemma 4.1: Let $T$ be a PDST for $S$ with root $v(q)$, where $q$ is a literal. Let $w[C]$ be a child of $v$. Then a PDST for $S\setminus\{q\}$ with top clause $C\setminus\{q\}$ (call it $T'$) may be formed as follows:

1. The root of $T'$ is $v'(\top)$;

2. The single subtree of $v'$ is a copy of the tree rooted at $w$, except the clause labels are strengthened by $\{q\}$.  

$S = \neg a, \neg b, \neg c \quad \neg a, \neg b, c \quad \neg a, b, \neg c \quad \neg a, b, c \quad a, \neg b, \neg c \quad a, \neg b, c$
Proof: No clause in the tree of \( w \) contains \( q \), so all clause labels of \( T' \) are well-defined. No goal in the tree of \( w \) is labeled by \( q \) or \( \neg q \), so all the goal nodes of \( T' \) have legal labels. Finally, the parent-child structure satisfies the definition of PDST.

**Lemma 4.2:** Let \( T \) be a PDST for \( S \) with top clause node \( w[C] \). Let \( v(q) \) be a child of \( w \). Then the tree rooted at \( v \) is a PDST for \( S \) with top goal \( q \).

*Proof:* Immediate from the definition.

**Corollary 4.3:** For a given formula \( S \) and a given top clause \( C \) or given top goal \( q \), the completed PDST is unique, up to reordering of children. Also, this is true for the universal PDST.

*Proof:* By induction on \( ||S|| \) (Definition 2.5), using Lemmas 4.1 and 4.2.

**Example 4.3:** Consider the PDST of Example 4.2 and Figure 5. The strengthening of \( S \) with \( \neg a \) yields

\[
S \neg a = \{[\neg b, \neg c], [\neg b, c], [b, \neg c] \}
\]

To illustrate Lemma 4.1, the completed PDST for \( S|\neg a \) with top clause \( [\neg b, \neg c] \) is essentially the left branch of Figure 5, with the goal \( \neg a \) replaced by \( \top \). Clauses containing literal \( a \) are replaced by their strengthened forms, producing the tree shown in Figure 6.

PDSTs do not generalize straightforwardly to the first order case. The reason is that a substitution must be applied throughout the derivation tree, not just to the subtree where the goal is unified with a clause. Consequently, in tableau search trees [LMG94], a search tree node is labeled with an *entire derivation tree*, as shown in Figure 7. In this example the five occurrences of the goal \( \neg a \) would in general be different due to differing substitutions in the various search nodes.

### 4.2 The PDST Game

The *PDST game* is a two-player game on a PDST over formula \( S \) with top clause \( C_{top} \) or top goal \( q \). Although many of the properties of the game hold for arbitrary PDSTs, they are of more interest on completed PDSTs or universal PDSTs. The play begins at the root and follows some branch of the PDST, ending at a leaf node. Alternately, one player, called the *refuter*, chooses a clause node and the other player, called the
spoiler, chooses a goal node. Each choice must be a child node of the other player’s prior choice. To begin play the refuter chooses some clause child of the root. The goal of the refuter is to end at a clause leaf. The goal of the spoiler is to end at a goal leaf. In other words, the player who cannot move loses. In Figures 4–6 the refuter’s choices correspond to dotted edges and the spoiler’s choices correspond to solid edges.

Informally, the refuter has a winning strategy for a particular PDST game if it is possible for the refuter to win that PDST game regardless of the choices made by the spoiler. The spoiler has a winning strategy in the dual situation. In Figure 4 the spoiler has a winning strategy by first choosing goal $\neg f$. The refuter must choose $[\neg b, \neg d, f]$, then the spoiler chooses goal $\neg b$. The formal definition follows. Essentially, the minimality criterion states that the refuter strategy is defined only at goal nodes that the spoiler would have an actual opportunity to choose, based on earlier choices in the same refuter strategy.

Definition 4.4: (winning refuter strategy) A winning refuter strategy is a partial function $\chi(v)$ from goal nodes to clauses nodes of the PDST upon which the game is being played such that:

1. $\chi$ is defined for the root of the PDST; if the root is $T$, then $\chi(T) = C_{\text{top}}$.
2. If $\chi(v)$ is defined and equals $w$, then $w$ is a clause child of $v$, and $\chi$ is defined on all children of $w$ (which may be the empty set).
3. $\chi(v)$ is undefined unless it is required to be defined by application of the above rules; that is, $\chi(v)$ is minimally defined.

Since it is impossible to define $\chi$ on a goal leaf, it is immediate that, if $\chi(v) = w$ and some child of $w$ is a failed goal node, then $\chi$ cannot be a winning refuter strategy.

Another view of a PDST is as an and-or tree, in which each clause node is an “or” node and each goal node is an “and” node. In Figures 4–6 the dotted edges descend from and-nodes. The literals labeling goal
nodes are immaterial for this evaluation. Goal leaves, being empty conjunctions, evaluate as \textit{true}, while clause leaves evaluate as \textit{false}. Recall that, if the boolean values are ordered as \textit{false} < \textit{true}, then “and” is “minimum” and “or” is “maximum”. Therefore, the refuter has a winning strategy if and only if the and-or tree evaluates to \textit{false}.

Some connections between the logical meaning of the PDST tree and the PDST game are given in the following theorems.

\textbf{Theorem 4.4:} Assume the refuter has a winning strategy in the above-described PDST game.

(A) If the root is \textit{T}, then \textit{S} is unsatisfiable.

(B) If the root is \textit{q}, then \textit{S} has no model in which \textit{q} is true.

\textit{Proof:} Let \(\chi(v)\) be a winning refuter strategy, according to Definition 4.4, and let \(W\) be the set of the nodes upon which \(\chi(v)\) is defined. In the PDST, prune away the subtrees rooted at all the unchosen clause nodes, i.e., those not in the image of \(\chi(v)\). What remains is a PDT, as every goal node has at most one child. A simple induction shows that every remaining goal node is in \(W\), and therefore has exactly one child. Therefore, according to Definition 3.2, this PDT is a refutation of \textit{S} in case (A) and is a refutation of \textit{q} w.r.t. \textit{S} in case (B). The theorem follows by Theorem 3.2.

\textbf{Theorem 4.5:} Given a PDST for \textit{S}, if \textit{S} has a model \(\mathcal{M}\) and the root \(\mathit{q}\) is in \(\mathcal{M}\) or the root is \textit{T}, then a winning strategy for the spoiler is always to choose a goal node in \(\mathcal{M}\).

\textit{Proof:} At clause node \(w[C]\), by construction of a PDST, each literal of \(C\) is a child if and only if it is not complemented in \(\text{ancs}(w[C])\). But by the strategy, every literal in \(\text{ancs}(w[C])\) is in the model \(\mathcal{M}\). Some literal of \(C\) is in \(\mathcal{M}\). Therefore \(w[C]\) has at least one child in \(\mathcal{M}\).

Notice that the converse of Theorem 4.4 does not hold, even for completed PDSTs. That is, in case (A) \textit{S} may be unsatisfiable but the refuter has no winning strategy that begins by choosing \(C_{\text{top}}\). Analogously, there is no model-elimination-style refutation with \(C_{\text{top}}\) as top clause. (Many other refinements of resolution also require one to know a clause or set of clauses in a minimal unsatisfiable set, to achieve completeness.) In case (B) \textit{S} may have no model in which \textit{q} is true, yet the refuter has no winning strategy on the completed PDST rooted at \(q\).

\textbf{Example 4.4:} The set of clauses in Figure 4 is actually unsatisfiable, so the spoiler cannot use a model as the basis of a winning strategy. However, for the top clause of the PDST in that figure the two goals, \(\neg f\) and \(\neg b\), encountered along the path chosen by the spoiler satisfy all of the clauses in which the variables \(b\) and \(f\) appear, either positively or negatively. By making choices according to the partial assignment, the spoiler never encounters a clause node lacking one of these goals as children. Thus the converse of Theorem 4.4, part (A) fails.

Similarly, the right subtree, rooted at \(\neg f\), is a completed PDST for \textit{S}, and \textit{S} has no model in which \(\neg f\) is true. However, the spoiler wins by choosing goal \(b\). Thus the converse of Theorem 4.4, part (B) fails.

\textbf{Theorem 4.6:}

...
Figure 8: A propositional derivation search tree (PDST) in which the spoiler chooses $\neg a$ in the left clause and chooses $a$ in the right clause. On the left, $\neg b$ is a failed goal node, while on the right $b$ is failed.

(A) If $S$ is a minimally unsatisfiable formula and $C_{top} \in S$, then a winning refuter strategy exists on any completed PDST for $S$ with top clause $C_{top}$.

(B) If $S$ has no model in which $q$ is true and $S$ has some model in which $\neg q$ is true, then a winning refuter strategy exists on any completed PDST for $S$ with top goal $q$.

(C) If $S$ is unsatisfiable, then a winning refuter strategy exists on the universal PDST for $S$.

Proof: For part (A) the proof is by induction on $|S|$. The basis, which is the empty clause, is immediate. For $|S| > 0$, let $q$ be any goal child of $C_{top}$. We need to show that the refuter has a winning choice at goal $q$. By Lemma 2.2, $S|q$ is unsatisfiable. Let $\mu(S|q)$ be any minimally unsatisfiable subset of $S$. The key observation of Anderson and Bledsoe is that $\mu(S|q)$ must contain some clause $C'$ that is a shortened version of $C \in S$. That is, $C' = C - \neg q$. Otherwise, $\mu(S|q)$ would be a proper subset of $S$, contradicting the hypothesis of minimality. It suffices for the refuter to choose $C'$. By the inductive hypothesis, a winning refuter strategy exists on the completed PDST for $\mu(S|q)$ with top clause $C'$. By Lemmas 4.1 and 4.2 this strategy transfers to the PDST for $S$ with top clause $C_{top}$, establishing part (A).

Part (B) follows easily from part (A) by adding the unit clause $[q]$ to $S$, then using it as top clause. Part (C) also follows from part (A), as the refuter can first choose any clause in $S$ in a universal PDST. 

The fact that one may have to begin proof searches from many top clauses to find a refutation partly explains why resolution-based methods have not been successful for high performance satisfiability checking. In some of the applications for satisfiability checking, a clause in a minimal unsatisfiable subset is not readily available.

Another problem is seen in the informal converse of Theorem 4.5, which also is not true. That is, when $S$ is satisfiable, there are, in general, winning strategies for the "spoiler" that do not always choose according to some fixed model $M$. Essentially, model elimination search procedures backtrack when they reach a failed node. But the failed nodes reached need not form a consistent set. Figure 8 illustrates this possibility.
The objective of the rest of this paper is to address these problems. We may think of satisfiability testing as evaluating the PDST game to see whether the refuter or spoiler has a winning strategy. The complete PDST conceptually defines the game tree, but it is not materialized. Instead, the procedure tries to explore just enough of the tree to determine the game outcome.

5 Autarkies

This section defines “autarky” and states the main results of the paper. The potential value of autarkies is suggested in Lemma 5.1, following the definition. Theorems 5.2 and 5.3 establish connections between autarkies and PDSTs. The application to refutation search efficiency is sketched at the end of the section and described in more detail in Section 6.

The concept of “autarky” was (to our knowledge) introduced into logic by Monien and Speckenmeyer, who called it “autark truth assignment”, employing the German adjective [MS85]. The word “autarky”, used mainly in economics, literally means “self-sufficient country or region”.

**Definition 5.1:** (autarky, autsat, autrem) Let \( S \) be a set of CNF clauses. A partial assignment \( M \) (Definition 2.2), possibly defined on some variables that do not occur in \( S \), is called an autarky of \( S \) if \( M \) partitions \( S \) into two disjoint sets,

\[
S = \text{autsat}(S, M) + \text{autrem}(S, M)
\]

such that each clause in \( \text{autsat}(S, M) \) is satisfied by \( M \) and each clause in \( \text{autrem}(S, M) \) has no variables in common with the variables that occur in \( M \). In particular, no literal of a clause in \( \text{autrem}(S, M) \) is complemented in \( M \). □

**Lemma 5.1:** Let \( M \) be an autarky of formula \( S \).

(A) \( S|M = \text{autrem}(S, M) \).

(B) If \( S \) is unsatisfiable, then \( \text{autrem}(S, M) \) is also unsatisfiable.

(C) If \( S \) is satisfiable, then \( M \) can be extended to a model of \( S \).

**Proof:** Part A is immediate from the definition. For parts B and C, observe that \( \text{autrem}(S, M) \) has no occurrence of any variable that occurs in \( M \). Suppose \( \text{autrem}(S, M) \) has a model \( M_{rem} \). Then \( M + M_{rem} \) is a model of (i.e., satisfies) \( S \). □

In the terminology of Monien and Speckenmeyer [MS85], a partial assignment is “autark in \( S \)” just when it is “an autarky for \( S \)” in our terminology. They describe a satisfiability algorithm that is a modification of the basic model searching algorithm of Davis, Logemann and Loveland [DLL62]. (The latter algorithm is often attributed incorrectly to Davis and Putnam [DP60]; we call this algorithm DPLL to acknowledge the contributions of all four authors). The modification consists of testing whether certain subsets of the literals in a shortest clause comprise an autarky before applying the DPLL splitting rule to a variable in that clause. They show that this modification guarantees fewer than \( 2^n \) splitting steps on a formula of \( n \) variables.

This paper investigates the application of the autarky concept to resolution-based methods. We begin by examining the relationship of autarkies to the PDST game of Section 4.2.

**Example 5.1:** As in Example 2.1, let

\[
S = \{[a, b], \neg a, c, b, d\}
\]
Figure 9: A completed propositional derivation search tree (PDST) with top goal $d$ for the same formula as Fig. 4.

Then \{a, c\} is an autarky of $S$, with

$$\text{autsat}(S, \{a, c\}) = \{[a, b], [-a, c]\}$$
$$\text{autrem}(S, \{a, c\}) = \{[b, d]\}$$

However, \{a\} is not an autarky because of clause $[-a, c]$. □

As seen in the previous example, another way to characterize an autarky $M$ is that $S|M \subseteq S$, that is, no clauses are shortened by the strengthening, although some clauses may be deleted.

The following theorems indicate how autarkies can interact with a refutation search. The first theorem shows that clauses satisfied by an autarky can be ignored, and the second shows how autarkies can be expanded by failed refutation searches. The actual algorithm appears in Section 6.

**Example 5.2:** The next theorem is illustrated by Example 4.1 and Figure 4. Let $M = \{-b, -f\}$, which is an autarky. We saw in Example 4.4 that the refuter has no winning strategy with the top clause $[b, -f]$, so the theorem holds in this case.

Now consider a different PDST for the same $S$, this time with top goal $d$, which has two clause children, $[-b, -d, f]$ and $[c, -a]$, as shown in Figure 9. The theorem asserts that the refuter cannot succeed by choosing the first clause, because $M$ satisfies it. Indeed, if the refuter did choose the first clause, then the spoiler would choose the goal that occurs in $M$, which is $-b$. The refuter’s next choice would necessarily be among clauses that were satisfied by $M$, in this case $[b, -f]$. The spoiler would choose the goal $-f$, winning the game. □

**Theorem 5.2:** Let $T$ be a completed PDST for formula $S$, and let $M$ be an autarky for $S$. Assume that the root of $T$ is labeled either with $\top$ or a literal $q$ such that $-q$ is not in $M$. If clause $C \in \text{autsat}(S, M)$, then $C$ is not chosen by any winning refuter strategy $\chi$ on $T$. 

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Proof: In this proof, the phrase “winning refuter strategy” is abbreviated to “strategy”. We give the proof for the case that the root of \( T \) is a literal \( q \); the proof for \( \neg q \) is similar. The proof is by induction on \( h \), the goal-height of \( T \).

The base case is goal-height zero (the root is the only goal node). The only possible clause in \( T \) is \( \neg q \), which is not in \( \text{autsat}(S, M) \), so the theorem holds vacuously.

For goal-height \( h > 0 \), assume the theorem holds for trees of goal-height less than \( h \). Let \( w[C_0] \) be any child of the root of \( T \), and let \( C_0 = [\neg q, p_1, \ldots, p_k] \), where \( k \geq 0 \). Then \( w \) has precisely \( k \) goal children, denoted as \( v_i(p_k) \).

First, suppose some strategy \( \chi \) chooses \( w \). For all \( 1 \leq i \leq k \), partial function \( \chi \) must be defined for \( v_i(p_k) \), so let \( \chi(t_i(p_k)) \). By the inductive hypothesis on the tree rooted at \( v_i(p_k) \), \( C_i \not\in \text{autsat}(S, M) \). But \( \neg p_k \in C_i \), so \( p_k \) (as well as \( \neg p_k \)) is not in \( M \). Also, \( \neg q \) is not in \( M \). Therefore, \( C_0 \) is not in \( \text{autsat}(S, M) \), which establishes the contrapositive of the theorem for the children of the root.

Now for \( 1 \leq i \leq k \), still supposing that strategy \( \chi \) chooses \( w \), the tree rooted at \( v_i(p_k) \) can be transformed into the completed PDST (call it \( T_i \)) for \( S \setminus \{q\} \) with top goal \( p_k \), as described in Lemmas 4.1 and 4.2. It is immediate that \( \chi \) defines a strategy on \( T_i \). But \( M \) is also an autarky for \( S \setminus \{q\} \). By the inductive hypothesis, no clause with a literal in \( M \) is chosen by \( \chi \). Therefore, \( \chi \) chooses no clause anywhere in the tree of \( w \).

Finally, let \( w[C_0] \) be a child of the root of \( T \) such that no strategy chooses \( w \). Then no strategy chooses a clause in the tree of \( w \), by minimality of strategies (Definition 4.4).

Example 5.3: The next theorem is also illustrated by Example 4.1 and Figure 4. This time, consider the completed PDST rooted at the goal \( \neg f \). Let \( M \) be the empty set, which is an autarky, and does not contain \( \neg f \) or \( f \). There is no winning refuter strategy, so the theorem asserts that \( M \) can be extended to some autarky \( M' \) that contains \( \neg f \). Simply adding \( \neg f \) is insufficient, due to \( \neg \neg b, \neg d, f \). However, adding both \( \neg f \) and \( \neg b \) makes \( M' \) an autarky.

Theorem 5.3: Let \( T \) be a completed PDST for formula \( S \) with root \( v(q) \), where \( q \) is a literal, such that there is no winning refuter strategy for \( T \). Let \( M \) be an autarky for \( S \) such that neither \( q \) nor \( \neg q \) is in \( M \). Then there is an autarky \( M' \supseteq M \) such that \( q \) (interpreted as a unit clause) is in \( M' \). Moreover, \( M' \setminus M \subseteq \text{hfs}(S) \).

Proof: In this proof, the phrase “winning refuter strategy” is abbreviated to “strategy”. The proof is by induction on the ordered pair \( (h, m) \), where \( h \) the goal-height of \( T \), and \( m \) is the number of children of the root \( v \) that are in \( \text{autrem}(S, M) \). The order is lexicographic on the pairs of integers.

The base case is goal-height \( h = 0 \) (the root is the only goal node). The hypotheses of the theorem fail for all \( m > 0 \). When \( m = 0 \), and no strategy exists, \( v(q) \) is a failed goal node, so no clause in \( S \) contains \( \neg q \). Therefore, \( M + q \) is also an autarky for \( S \).

For goal-height \( h > 0 \), assume the theorem holds for all trees characterized by \( (h', m') \), where \( (0, 0) \leq (h', m') < (h, m) \) in lexicographic order. Let the children of \( v \) that are in \( \text{autrem}(S, M) \) be \( w[C_i] \), for \( 1 \leq j \leq m \). If \( m = 0 \), then every clause of \( S \) that contains \( \neg q \) also contains a literal in \( M \), so again, \( M + q \) is also an autarky for \( S \). If \( m > 0 \), we will first construct \( M_1 \supseteq M \), such that \( C_0 \in \text{autsat}(S, M) \) (and neither \( q \) nor \( \neg q \) is in \( M_1 \), and \( M_1 \setminus M \subseteq \text{hfs}(S) \)). Then \( v \) has at most \( m - 1 \) children that are in \( \text{autrem}(S, M_1) \), so the required \( M' \supseteq M_1 \) exists by the inductive hypothesis.

To complete the proof, we need to construct \( M_1 \). Let the children of \( w \) be \( v_i(p_k) \), and for each \( v_i(p_k) \) consider the completed PDST for \( S \setminus \{q\} \) with top goal \( p_k \). Call it \( T_i \). If every \( T_i \) had a strategy, these could be combined in \( T \) to produce a strategy that chooses \( w \), using Lemmas 4.1 and 4.2. Therefore, some \( T_i \) has no strategy, and is of goal height \( (h - 1) \). Also, \( M \) is an autarky for \( S \setminus \{q\} \).

1. If \( \neg p_k \) is not in \( M \), then the inductive hypothesis implies that an autarky \( M_1 \supseteq M \) exists and contains \( p_k \). \( M_1 \) contains neither \( q \) nor \( \neg q \) because they do not appear in \( S \setminus \{q\} \).
2. If \( \lnot p_k \in M \), then some other literal of \( C^{(1)} \), say \( r \), must be in \( M \) by the definition of an autarky, so let \( M_1 = M \) in this case.

In both cases, \( C^{(1)} \in \text{autsat}(M_1) \), as required. 

5.1 Propositional Application for Autarky Analysis

Examining the proof of Theorem 5.3, we arrive at the following idea for autarky construction during a refutation search. Section 6 develops an algorithm in detail.

1. When the search for a refutation of a specific goal \( q \) begins, an “initial autarky” \( M_0 \) (possibly \( \emptyset \)) is passed in. Assume that neither \( q \) nor \( \lnot q \) is in \( M_0 \).

2. For each clause \( C^{(j)} \) \( (1 \leq j \leq k) \) that is eligible for extension (i.e., contains \( \lnot q \) and does not contain any ancestor), if the “current autarky” \( M_{j-1} \) satisfies \( C^{(j)} \), then \( C^{(j)} \) is bypassed, and \( M_j = M_{j-1} \).

3. For each clause \( C^{(j)} \) \( (1 \leq j \leq k) \) that is tried as an extension, a “current autarky” \( M_{j-1} \) is passed down into a recursive search.

If the extension fails to lead to a refutation, then the recursive procedure passes back up an “increment” \( \Delta M_j \), and a new “current autarky” \( M_j = M_{j-1} + \Delta M_j \) is computed. The increment is supplied by a goal child of \( C^{(j)} \) that could not be refuted.

4. If all clauses fail to lead to a refutation of \( q \), then the “final autarky increment” is \( \Delta M_{\text{final}} = \sum \Delta M_j + q \).

5. If any clause leads to a refutation of \( q \), then the “final autarky increment” is \( \emptyset \).

6. Pass back the “final autarky increment”.

Of course, the refutation search passes around other information, as well; this outline just mentions that related to autarky computation.

Example 5.4: Again consider Example 4.1 and Figure 4. With \( [b, \lnot f] \) as the initial top clause, a refutation procedure (selecting literals depth-first, left-right) would refute goal \( b \) at level 1, then search for a refutation of \( \lnot f \) at level 1. This leads to an extension, then to the failed goal \( \lnot b \) at level 2. Thus \( \lnot b \) is passed back to level 1 as the final autarky increment. Back at level 1, there are no more clauses to try, so the autarky increment from level 2 is combined with this goal, and passed up to level 0 as \( \{\lnot b, \lnot f\} \). At the top level we had \( M_0 = \emptyset \), so \( M_1 = \{\lnot b, \lnot f\} \). This is an autarky for the entire formula \( S \).

Now we know that \( S \) is unsatisfiable if and only if \( \text{autrem}(S, M_1) \) is. In other words, clauses \( [\lnot b, \lnot d, f] \) and \( [c, \lnot f] \) do not need to be considered as alternate top clauses for new refutation attempts. Since \( M_1 \) is now the “current autarky” at level 0, the procedure sketched above bypasses them. In this example, any new top clause selected from \( \text{autrem}(S, M_1) \) leads to a successful refutation. In general, the next top clause might also fail, and \( M_1 \) would be expanded to a larger autarky \( M_2 \), etc. 

5.2 First-Order Application for Autarky Analysis

Exploiting autarkies in the first-order case is considerably more complicated, and requires considerable further study. This section will present some possibilities by means of an example in first-order logic without equality.
Example 5.5: Let capital letters denote first-order variables, as in Prolog, for this example. Formula $S_{f,a}$ consists of:

\[
\begin{align*}
&\neg p(W,X,Y), \neg s(W,X,Y) \\
&p(W,X,Y), \neg q(W,X,Z), \neg r(X,Y,Z) \\
&q(W,W,Z)
\end{align*}
\]

Let the top clause be $[\neg p(W,X,Y), \neg s(W,X,Y)]$. Assume the search tries clauses from top to bottom and selects literals from left to right within the clause. Nodes in the first-order search tree are labeled $\tau_0, \tau_1, \ldots$, in prefix order, to facilitate discussion.

On the left, Figure 10 shows the first-order search tree after two extensions. In node $\tau_3$, $\neg q(W,W,Z)$ has been refuted and the resulting C-literal (see Section 7) has been attached. Literal $\neg r(W,Y,Z)$ is selected next.

\[
W = a; Y = b; Z = f(a)
\]
On the right of Figure 10 we see the continuation of the search tree after two more extensions from \( \sigma_1 \). Node \( \sigma_1 \) is the first point of failure. Nodes \( \sigma_2 \) and \( \sigma_3 \) have no alternatives, so the search now backtracks to node \( \sigma_1 \). In particular, these goals have failed: \( s(f(a), b, a), p(f(a), b, a), \neg r(W, Y, Z), \neg p(W, W, Y) \).

Let us now review briefly the idea of anti-lemmas [LMG84]. Subgoal \( \neg q(W, W, Z) \) was successfully refuted in search node \( \sigma_1 \), and the corresponding C-literal was introduced. Now \( \sigma_2 \) has been backtracked over, due to a later failure. However, the C-literal is now attached to the goal \( \neg q(W, X, Z) \) in search node \( \sigma_1 \) as an anti-lemma (see Figure 11). The next clause that resolves with \( \neg q(W, X, Z) \) (the selected literal of search node \( \sigma_1 \)) is \( [q(a, a, Z)] \), leading to node \( \sigma_5 \). But \( q(a, a, Z) \) is an instance of the anti-lemma, so this line of derivation must eventually fail also. Therefore, node \( \sigma_5 \) may be abandoned immediately.

The next clause that resolves with \( \neg q(W, X, Z) \), the selected literal of search node \( \sigma_1 \), is \( [q(b, a, f(a)))] \), leading to node \( \sigma_6 \) (see Figure 11). This is not an instance of the anti-lemma. However, observe that literal \( \neg r(a, Y, f(a)) \) in \( \sigma_6 \) is an instance of the literal \( \neg r(W, Y, Z) \), which was found not to be refutable in an earlier branch of the search. We claim that a refutation of \( \neg r(a, Y, f(a)) \) cannot exist. Based on this claim, the node \( \sigma_6 \) may be abandoned also.

Figure 11: A first-order search tree, developed from Fig. 10, in which the search may be abandoned at nodes \( \sigma_5 \) and \( \sigma_6 \) for reasons discussed in Example 5.5.
The reasoning to support the claim runs as follows. Suppose a first-order refutation of \( \neg r(a, Y, f(a)) \) could be completed in this tree. Then it could be further instantiated, if necessary, into a ground (variable-free) refutation. Consider the same substitutions in the branch of the search that failed (and possibly some extras to force groundedness). Collecting all the ground clauses from the hypothetical successful refutation and the failed branch, we have a finite set of propositional clauses, \( S \), even though the Herbrand universe is infinite. Recall that \( \operatorname{hits}(S) \) denotes the set of literals that occur in \( S \).

There is an autarky for \( S \) that contains all the instances of \( \neg r(W, Y, Z) \). Specifically, it is the set \( M \) containing all literals in \( \operatorname{hits}(S) \) that are instances of \( s(f(a), b, a), p(f(a), b, a), \neg r(W, Y, Z), \) or \( \neg p(W, W, Y) \). It is easy to see that \( M \) is consistent (interpreted in the Herbrand universe, so that \( a, b \), and \( f(a) \) are distinct). To verify that \( M \) must be an autarky, it is sufficient to check for each clause of \( S_f.a. \) that any instance that has a literal that is complementary to a literal of \( M \) also has another literal that is in \( M \).

Consequently, no refutation of any instance of \( \neg r(a, Y, f(a)) \) can exist in this set of propositional clauses, by Theorem 5.2.

We emphasize that the above reasoning applies to this specific example, but we are not prepared to formulate a general statement. \( \square \)

We believe that this example indicates a possible approach for applying the autarky concept to first-order theorem proving. However, even if the idea can be developed into a general tool, practical experience is needed to discover whether the overhead incurred in autarky analysis pays sufficient dividends in terms of pruning fruitless searches.

6 The Modoc Algorithm

The main procedures of the *Modoc* algorithm are shown in Figures 12 and 13. The top level is shown in Figure 14. Autarky processing is carried out by the subroutines \texttt{inAutRem}, \texttt{addAutarky}, and \texttt{delAutarky}. Other subroutines implement propositional model elimination, or equivalently, a search of the universal PDST (in mode \texttt{ALL}) or the PDST with a user-specified top clause (in mode \texttt{SINGLE}).

The syntax is an amalgam of C, Pascal and ML, with a bit of overloading thrown in, which should be mostly self-explanatory. Recall Definition 2.3 for “+” and “−” on sets. One notation that might need explanation is

\[
(x, y, z) := f(u, v);
\]

In the spirit of ML, this denotes that function \( f \) returns a *tuple* of 3 values. These three values are then assigned respectively to \( x \), \( y \) and \( z \).

We assume an abstract data type \texttt{Context} that maintains implementation-dependent information about the state of the refutation search. The variable \( F \), of type \texttt{Context}, is a reference, or pointer, to some structure containing this information. Procedures with \( F \) as parameter may change the context state, while *functions* do not. Procedures and functions can be differentiated by the convention that procedures do not return values. The exception is the “constructor function” \texttt{setupContext}, which initiates a context with no ancestors, no lemmas, and an empty autarky, and returns a reference (pointer) to it.

**Definition 6.1:** (context) On a conceptual level a \texttt{Context} may be thought of as having four components. If \( F \) is of type \texttt{Context}, then these fields are denoted as follows for this discussion:

- clauses \( F \rightarrow S \)
- ancestors \( F \rightarrow A \)
- lemmas \( F \rightarrow L \)
- autarky \( F \rightarrow M \)
Where the meaning is clear, the prefix “F →” may be omitted.

The invariant is maintained that the set of “autarky literals” $F \rightarrow M$ is actually an autarky for $S \mid A$, but not for $S$ itself.

The handling of model-elimination lemmas is not an integral part of the Modoc algorithm. The pseudo-code indicates where lemma-related processing would probably occur, but many variations are possible. Section 7 discusses lemmas in the prototype implementation, primarily to support the interpretation of the experimental results.

Ancestors are maintained as a stack. Each ancestor has a set of lemmas associated with it, and a set of “autarky literals” associated with it. Of course, either set might be empty. The association is actually with the depth of the ancestor rather than its literal. Procedures that manipulate the state of the context are now described.

**pushAncestor, popAncestor:** The obvious stack operations on $F \rightarrow A$; the depth is explicit only for readability.

**inAutRem(F, curClause):** Returns true if *curClause* has no literal that is in $F \rightarrow M$.

**delAutarky(F, curM):** Removes the literals of *curM* from $F \rightarrow M$. The set *curM* should be identical to the subset of $F \rightarrow M$ that was added during failed attempts to refute the current goal $q$. The procedure is called when a refutation of $q$ has been found.

**addAutarky(F, q, depth):** Add $q$ to the subset of $F \rightarrow M$ associated with depth. The procedure is called when all attempts to refute $q$ have failed. The depth parameter is strictly unnecessary, but is used to aid the checking of internal consistency.

**screenResolvables(F, q):** Assuming $q \neq \top$, this function finds all clauses containing $\neg q$, then eliminates those having a literal in $F \rightarrow A$ and returns the rest in an ordered list. For efficiency, the returned list begins with clauses having 0 subgoals, if any, followed by clauses having 1 subgoal, followed by all others.

If $q = \top$ and mode = ALL, the returned list contains all clauses of $S$, with the user-specified initial top clause first in the list.

If $q = \top$ and mode = SINGLE, the returned list contains just the user-specified initial top clause.

**extractSubgoals(F, lits):** Deletes from lits those literals that are complementary to $(A \cup C)$. Those remaining are partitioned into failed goals and goals with at least one clause eligible for extension.

### 6.1 Correctness

This section addresses correctness of *Modoc* without lemmas. Soundness of lemma processing is an orthogonal issue, and depends on exactly how lemmas are added to the basic algorithm. The main idea is that *Modoc* simply evaluates the universal PDST for the formula $S$ when mode equals ALL, or the PDST for a user-specified top clause when mode equals SINGLE (see Figure 14). A call to tryRefuteSubgoal positions the search at a goal node, and evaluates the refuter’s choices of clause. A choice that returns UNSAT is a winning choice for the refuter, so if one such is found, additional choices need not be evaluated. A call to tryRefuteClause positions the search at a clause node, and evaluates the spoiler’s choices of goal. Now, a
tryRefuteSubgoal(F:Context, q:Literal, depth:integer) : (Status, LitSet, ClauseList)
/* Returns (result, ΔM, ΔR). result = SAT or UNSAT.
** If SAT, ΔM holds autarky Lits. If UNSAT, ΔR holds refutation. */
begin
ancIsLemma := isLemma(F, q);
if (ancIsLemma)
    hideLemma(F, q);
pushAncestor(F, q, depth);
assert(not inAutarky(F, invertLit(q)); /* When loop exits, result = SAT or UNSAT. If SAT, curM holds autarky Lits.
** If UNSAT, curClause holds the refuted clause id. */
remClauses := screenResolvables(F, q);
result := SAT;
curClause := 0;
curM := ∅;
while (remClauses ≠ ∅ and result == SAT)
begin
    curClause := getMember(remClauses);
    remClauses := remClauses - curClause;
    if (inAutRem(F, curClause))
        begin
            (result, ΔM1, ΔR) := tryRefuteClause(F, curClause, depth + 1);
            curM := curM + ΔM1;
            /* If result = UNSAT, loop will exit. */
            end
    end
if (result == UNSAT)
begin
    addLemma(F, invertLit(q), curClause, depth);
delAutarky(F, curM);
ΔM := ∅;
end
else
begin
    addAutarky(F, q, depth);
    ΔM := curM + q;
    ΔR := ∅;
    end
if (ancIsLemma)
    exposeLemma(F, q);
delLemmas(F, depth);
popAncestor(F, q, depth);
return (result, ΔM, ΔR);
end

Figure 12: Procedure tryRefuteSubgoal of Modoc Algorithm. Two mutually recursive procedures comprise the reasoning engine. See text for discussion.
tryRefuteClause(F:Context, curClause:Clause, depth:integer) : (Status, LitSet, ClauseList)
/* returns (result, ΔM, ΔR). result = SAT or UNSAT.
** If SAT, ΔM holds autarky Lits. If UNSAT, ΔR holds refutation. */
begin
lits := getClauseLits(curClause);
(failedSubgoals, goodSubgoals) := extractSubgoals(F, lits);
if (failedSubgoals ≠ ∅)
begin
addAutarky(F, failedSubgoals);
ΔM := failedSubgoals;
ΔR := ∅;
result := SAT;
return (result, ΔM, ΔR);
end
result := UNSAT;
remSubgoals := goodSubgoals;
while (result == UNSAT and remSubgoals ≠ ∅)
begin
q := getMember(remSubgoals);
remSubgoals := remSubgoals - q;
if (not isLemma(F, invertLit(q)))
begin
(result, ΔM, ΔR1) := tryRefuteSubgoal(F, q, depth);
ΔR := append(ΔR, ΔR1);
/* If result = SAT, loop will exit. */
end
end
if (result == UNSAT)
begin
ΔM := ∅;
ΔR := append(list(curClause), ΔR);
end
return (result, ΔM, ΔR);
end

Figure 13: Procedure tryRefuteClause of Modoc Algorithm. Two mutually recursive procedures comprise
the reasoning engine. See text for discussion.

choice that returns SAT is a winning choice for the spoiler, so if one such is found, additional choices need not
be evaluated. However, knowledge of autarkies is used to avoid searches where the outcome is predictable.
The stack of alternating goals and clause-ids determines a position in the PDST or universal PDST
that the search has reached upon an invocation of tryRefuteSubgoal or tryRefuteClause. The inductive
hypothesis is that the algorithm behaves correctly on proper subtrees of the tree rooted at this point.
The notation of Definition 6.1 is used. However, in this section, the set of lemmas F → L remains empty.
Recall the definition S|A from Definition 2.4.
modoc(S:Formula, initTopClause:Clause, allTop:integer) : (Status, LitSet)
/* returns (result, M, R).
** result = SAT or UNSAT. If SAT, M holds autarky Lits.
** In this case, if mode was ALL, M is a model of S;
** otherwise, initTopClause is satisfied by M, but S may not be.
** If UNSAT, R holds refutation. */
begin
  if (allTop /== 1) mode := ALL; else mode := SINGLE;
  F := setupContext(S, initTopClause, mode);
  (result, M, R) := tryRefuteSubgoal(F, T, 0);
  return (result, M, R);
end

Figure 14: Top level of Modoc Algorithm. In mode ALL, all clauses are eligible as top clauses, but the search tries initTopClause first. Either a refutation is found or a model is found. In mode SINGLE, either a refutation for initTopClause specifically is found or an autarky is found that satisfies that clause. No clause satisfied by the autarky is part of a minimal unsatisfiable set of clauses. See text for discussion.

If tryRefuteSubgoal returns UNSAT, then S|A+q is unsatisfiable. If tryRefuteClause returns UNSAT, then S|A is unsatisfiable. These facts follow from the analogy with PDST trees (Theorem 4.4), and will not be re-proved. As pointed out by Shostack, a list of clauses in the order in which they are used for extensions suffices to specify a propositional model-elimination derivation [Sho76]. It is straightforward that the algorithm returns such a list when it returns UNSAT. Also, by simple inspection of the algorithm, when either function returns UNSAT, then F → M is unchanged upon exit from the value upon function invocation. This follows in tryRefuteSubgoal because the call to delAutarky deletes any literals added to F → M since the procedure invocation.

Theorem 6.1: Let F denote the context. Let A denote F → A upon a function invocation. Let M₀ denote F → M upon a function invocation.

(A) Suppose tryRefuteSubgoal(F, q, depth) is called. If M₀ is an autarky for S|A and contains neither q nor ¬q, and tryRefuteSubgoal returns (SAT, ΔM, ΔR), then upon exit:
1. F → A = A upon exit.
2. q ∈ ΔM,
3. M₀ + ΔM is an autarky for S|A,
4. M₀ + ΔM is the value of F → M upon exit.

(B) Suppose tryRefuteClause(F, curClause, depth) is called. If M₀ is an autarky for S|A and does not satisfy curClause, and tryRefuteClause returns (SAT, ΔM, ΔR), then upon exit:
1. F → A = A upon exit.
2. ΔM satisfies curClause,
3. M₀ + ΔM is an autarky for S|A,
4. M₀ + ΔM is the value of F → M upon exit.
Proof: Consider first $\text{tryRefuteSubgoal}$. By hypothesis, $q \not\in M_0$, so $M_0$ is also an autarky for $S|(A+q)$. Let us define

$$M_i = M_0 + \sum_{j=1}^{i} \Delta M_j$$

where $\Delta M_j$ is returned by the $j$-th recursive call of $\text{tryRefuteClause}$. (Note that cur$\mathcal{M}$ takes on successive values of $\sum_{j=1}^{i} \Delta M_j$.) It follows that the preconditions of $\text{tryRefuteClause}$ hold at the time of the recursive call. By Theorem 5.3 (and the assumed correctness of the recursive call), $M_i$ is an autarky for $S|(A+q)$. The calls to $\text{addAutarky}$ keep $F \rightarrow M$ equal to $M_i$ as defined here, so any clauses bypassed by the test $\text{inAutRem}$ must evaluate to SAT with respect to the $M_i$ in effect when the clause was bypassed. But $\Delta M = \text{cur}\mathcal{M}_{\text{after}} + q$. Thus every clause containing $\neg q$ contains a literal in $A$ or in $M_0 + \Delta M$. This is also true for clauses containing $q$, so Part A is established.

For Part B, we only need to observe that $\text{tryRefuteClause}$ returns SAT only if a subgoal with no possible extensions is found, or some recursive call to $\text{tryRefuteSubgoal}$ returns SAT. In the first case correctness is obvious. In the second case, all prior recursive calls to $\text{tryRefuteSubgoal}$ returned UNSAT, and did not change $F \rightarrow M$. By the inductive hypothesis, $F \rightarrow M$ is an autarky as of the exit from the recursive $\text{tryRefuteSubgoal}$ that returned SAT, and clearly it satisfies cur$\mathcal{M}$.

Corollary 6.2: If Modoc operates in mode ALL, and returns SAT, then $F \rightarrow M$ is a model of $S$.

Proof: Similar to Part A of Theorem 6.1, with $A = \emptyset$, $M_0 = \emptyset$. In the top-level invocation of $\text{tryRefuteSubgoal}$ a list of all clauses in $S$ is returned by $\text{screenResolvables}$. As each clause is considered as $\text{cur}\mathcal{M}$ in the loop, either it is already satisfied by $F \rightarrow M$ and bypassed, or a refutation is attempted and fails. In the latter case, the clause is satisfied by $F \rightarrow M$ after $\text{tryRefuteClause}$ returns. But $F \rightarrow M$ grows monotonically, so upon exit $F \rightarrow M$ satisfies every clause in $S$.

Note: Although the mode ALL guarantees a definitive result for both Modoc and model elimination, all of the experiments reported were run in mode SINGLE to avoid possible excessive time by model elimination on satisfiable formulas.

## 7 Lemmas and C-Literals

In the model elimination procedure a “lemma” may be recorded upon the completion of any (sub)refutation [Lov69, FLSY74, Lov78]. This is a clause that is logically implied by the original formula. Lemmas are not necessary for completeness of model elimination. Shostack proposed a “C-literal” mechanism that is reasonably efficient to implement [Sho74, Sho76]. This approach has been extended by Letz et al., who give a detailed treatment and propose a related pruning strategy, called “strong regularity” [LMG94]. A detailed treatment of lemma strategies is beyond the scope of this paper. This section sketches how lemmas are incorporated into the prototype implementation of the Modoc algorithm. We introduce and describe a strategy for “quasi-persistent” lemmas.

In general, the trade-offs are not well understood between the cost of storing and maintaining the lemma (considering it for resolutions, etc.), vs. the time saved when it can be used. This topic has been studied empirically for first order theorem proving. Although initial experience was negative [FLSY74], subsequent reports were more positive [AS92, Shi94, LMG94]. To the best of our knowledge, there are no empirical studies of lemma strategies on propositional problems.

Suppose the refutation of a literal $q$ is completed at a point in a PDT (Definition 3.2) where the set of proper ancestors of $q$ is $A$. Let $B$ be the subset of $A$ that represents the goals that were actually used in the refutation (say $B = \{p_1, \ldots, p_m\}$, where $m$ may be 0). In model-elimination terms they were used...
for reductions; in PDT terms the $p_i$ *prevented* the goals $\neg p_i$ from appearing in the subtree below $q$. Then a lemma clause, $[\neg q, \neg p_1, \ldots, \neg p_n]$, can be derived soundly [LMG94]. This is an implication of the form $(B \supset \neg q)$; Loveland described the same inference, but using quite different terminology, such as scope of A-literals [Loveland].

**Example 7.1:** Recall the refutation search described in Example 5.4, based on Figure 4. To refute $e$, the goal is extended with the clause $[\neg e, \neg e]$, which has no subgoals, due to the ancestor $e$. Therefore, the lemma $[\neg e, \neg e]$ follows. However, this is already a clause in the formula.

But this also completes the refutation of $e$. No proper ancestors of $e$ were used for reductions, so the lemma $[\neg e]$ follows.

Similarly, the refutation of goal $\neg d$ is now complete. This refutation used goal $e$ for reduction, but $e$ is beneath $\neg d$ in the tree, so is not part of the lemma. The lemma is simply $[d]$. \(\square\)

Shostack proposed an efficient strategy to maintain such lemmas in model elimination chains [Sho76]; Letz et al. generalized it to trees [LMG94]. In the lemma $[\neg q, \neg p_1, \ldots, \neg p_n]$, literal $\neg q$ is called a “C-literal” and is attached to the *lowest* ancestor among $\{p_1, \ldots, p_n\}$; call this goal $p_c$. (If $m = 0$, attach it to the root of the PDT, which is normally $\top$.) This technique implicitly weakens the lemma to $(\text{ance}(p_c) \supset \neg q)$, because the precise set $\{p_1, \ldots, p_n\}$ is not recorded.

The lemma can only be used in the subtree of $p_c$, and in this context its only use is to “extend” another occurrence of the goal $q$, because subgoals $p_i$, being in $B$, cannot occur beneath $p_c$. Moreover, when the lemma is used, its clause node has no subgoals in the PDT. (In model elimination, the extension is followed immediately by reductions on all of the subgoals.) In this sense, a C-literal is somewhat like an ancestor, in that it immediately closes off a branch of the PDT tree, and the operation is sometimes called “C-reduction”.

If the PDT is abandoned (because some other part of the refutation fails) then the lemma is forgotten. If the (sub)refutation of $p_c$ is completed, the lemma is also forgotten in the sense that it is not used later in other (sub)refutations. Because of the limited application and lifetime of the lemma, it is actually unnecessary to record it fully. The C-literal $\neg q$ and the lowest ancestor $p_c$ are all that are needed. Thus the difference between $B$ and $\text{ance}(p_c)$ is immaterial with this strategy.

The main idea of *Modoc* is autarky pruning, which is compatible with the use of lemmas, and largely orthogonal. Clauses that are pruned by an autarky cannot participate in a successful refutation (at the point where they are pruned), whether or not lemmas are used to shorten the refutation. However, lemmas seem to be important for efficiency, so they were incorporated into the prototype implementation of *Modoc*.

### 7.1 Quasi-Persistent Lemmas

Our strategy varies from the C-literal strategy described above in that lemmas derived during failed (sub)refutations are not necessarily forgotten. Normally, a PDT is not completely abandoned, but only the subtree where the refutation fails is abandoned. (In the first order case, substitutions need to be backed out, as well.) The lemma can function as a C-literal until the subtree rooted at $p_c$ is abandoned, or the refutation of $p_c$ is completed (where $p_c$ and other terminology is continued from the previous subsection).

*Modoc* maintains lemmas attached at $p_c$ until the tree rooted there is abandoned or its refutation is completed. The previously described strategy of Letz et al. effectively deletes the lemma as soon as the refutation of any clause ancestor of $q$ fails. There are pros and cons of both strategies. Our strategy makes it unnecessary to re-derive the same lemma at the same attachment point so often, but it makes it necessary to record the full lemma.
Our strategy is incompatible with the heuristic called "strong regularity", introduced by Letz et al. That is, Modoc may undertake to refute a goal \( \neg q \) in the subtree (rooted at \( p_c \)) where \( \neg q \) is attached as a lemma. The "strong regularity" heuristic consists of avoiding such attempts. "Strong regularity" was shown to be complete under certain conditions, but quasi-persistent lemmas do not meet those conditions, and a counter-example can be constructed if the two heuristics are combined.

**Example 7.2:** This example continues the refutation search begun in Example 7.1, based on Figure 4. While refuting \( b \) the procedure would be able to attach C-literals \( \neg c, \neg f, \) and \( \neg b \) at the root. When the refutation fails in the right branch, the traditional C-literal technique forgets all of them. Our quasi-persistent method does not, because they are still sound as C-literals. When the refutation search tries a different top clause, the C-literals \( \neg c \) and \( d \) are available and might shorten the search. (In fact, \( [c, \neg d] \) now succeeds immediately.) This can also happen without backtracking to the top level. □

The quasi-persistent heuristic holds lemmas longer, but spends more time per lemma in bookkeeping, compared to the traditional C-literal method. There is no apparent way to determine which method performs better except empirical testing.

### 7.2 Lemma-Induced Cuts

We now describe the method by which Modoc exploits complementary C-literals. Suppose, as described above, the C-literal \( \neg q \) is attached at \( p_c \) and the goal \( \neg q \) occurs in a subtree of \( p_c \). Should the refutation of \( \neg q \) be successful, there results a new lemma whose C-literal is \( q \). Now complementary C-literals have been derived on one branch. Let the full form of the second lemma be \([q, \neg r_1, \ldots, \neg r_n] \); that is, \( \{r_1, \ldots, r_n\} \) is exactly the set of ancestors used in the refutation of \( \neg q \). Again, let \( r_c \) be the lowest ancestor among the \( r_i \), or the root T if \( n = 0 \).

Now consider the lower of the two goal nodes \( p_c \) and \( r_c \). The situation is symmetric, so let us suppose it is \( r_c \). Let \( A' \) be the ancestors of \( r_c \). Now add a "virtual clause" \( \neg r_c, q, \neg q \) to the formula; this is a tautology, so it is harmless. However, extending \( r_c \) with this virtual clause creates goals \( q \) and \( \neg q \), both of which are immediately closed by the lemmas. Thus the goal \( r_c \) is immediately refuted, even though the tree in which the lemmas \( q \) and \( \neg q \) were derived is never completed to a refutation.

Introduction of the "virtual clause" described above is essentially a form of the cut rule [LMG94]. If \( S \) is the original set of clauses, we have discovered \((S + A') \vdash q \) and \((S + A') \vdash \neg q \). Now the cut rule infers \((S + A') \vdash \emptyset \).

While the introduction of a tautologous clause is always sound, it normally is not practical because the prover has no way to anticipate that each of the complementary literals has a short refutation. However, if a pair of complementary C-literals have been derived, then the prover has that information in hand.

This methodology also can be applied to first order proofs where the prover is not using strong regularity. In this case, a most general unifier of the complementary C-literals would be applied before creating the "virtual clause".

### 8 Experimental Results

A prototype implementation of Modoc was programmed in Prolog. This section reports on preliminary tests. The count of extension operations (essentially resolutions with an input clause) is reported. Prolog is not an efficient language for this type of algorithm, because of the large numbers of asserts and retracts that would simply be array accesses in an imperative language. Therefore, CPU times are not very meaningful.
Figure 15: Comparative Performances on random 3CNF formulas of 100 variables and 427 clauses (with initial pure literals eliminated by preprocessing). Operations reported: extensions for resolution methods and assignments (including unit and pure literals) for DPLL.

As one example, formula 111128, which proved the most difficult, took 111 CPU minutes on a Sun Sparc 10/41 for Modoc, while the DPLL program, coded in C, took 0.22 minutes for about the same number of basic operations. Nevertheless, these experiments show some definite trends.

The model elimination algorithm was obtained by disabling the autarky pruning from Modoc. All other techniques and heuristics are the same, such as the lemma strategy. Therefore, it seems reasonable to attribute the performance differences to autarky pruning.

For all tests the mode was SINGLE (see Figure 14) for Modoc and model elimination, so a refutation was attempted from one top clause only on each run reported.

Preliminary results on random 3CNF formulas are shown in Figure 15. These formulas were generated according to the constant-clause-width model: every clause containing 3 different variables and any combination of literal polarities is equally likely. The first clause is arbitrary chosen as the top clause.

Recall that Figure 1 showed that model elimination suffered a rapid performance degradation on small satisfiable random formulas, ranging from 10 to 15 variables, and became hopeless for 20 variables. Modoc overcomes that problem, and generally solves satisfiable formulas faster than comparable unsatisfiable formulas. In the range of 10 to 20 variables (no table shown), the differences between satisfiable and unsatisfiable formulas are minor; all were solved with 32–168 extensions, which agrees closely with the numbers for unsatisfiable formulas in Figure 15. At the level of 100 variables, Figure 15 shows that satisfiable formulas now exhibit significantly less difficulty for Modoc than unsatisfiable formulas. Also, Modoc sometimes accomplishes significant savings on unsatisfiable formulas, compared to model elimination without autarky pruning. Furthermore, a basic DPLL model-search algorithm used a number of operations that is the same order of magnitude for random formulas, although its basic operation is variable assignment followed by strengthening (Definition 2.4).

Figure 16 shows results on pigeon-hole formulas and modified pigeon-hole formulas. Recall that the $k$-pigeon problem is the same as the problem of coloring a $k$-clique graph with $k - 1$ colors. Due to their regular structure, there is no backtracking from a failed refutation in either Modoc or model elimination.  

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
Formula & Status & Modoc & Model Elim. & DPLL \\
& & Extensions & Extensions & Assignments \\
\hline
111120 & sat & 79,608 & ?? & 48,103 \\
111121 & sat & 18,586 & ?? & 126,206 \\
111122 & unsat & 336,316 & 1,626,291 & 254,075 \\
111123 & unsat & 298,091 & 1,304,025 & 154,759 \\
111124 & unsat & 253,256 & 255,968 & 94,076 \\
111125 & sat & 21,274 & ?? & 24,958 \\
111126 & unsat & 190,550 & 193,225 & 102,205 \\
111127 & sat & 77,202 & ?? & 59,399 \\
111128 & unsat & 360,949 & > 7,220,085 & 353,995 \\
111129 & sat & 26,869 & ?? & 3,930 \\
\hline
\end{tabular}
\end{table}

1Negative binary clauses encode the constraints that no two pigeons can occupy the same hole (no two nodes can have the same color), and positive $(k-1)$-clauses encode the constraints that each pigeon must be in some hole (each node must have some color).
The “current autarky” remains at the empty set on such formulas.

Satisfiable versions of pigeon-hole formulas were created by removing either the first or last binary clause. Observe the great performance differences, depending on which clause was removed, for each of the methods tested. These differences seem to be accidents of the way in which the formula is presented and stored. The main themes are seen again, but not so pronouncedly as for random formulas. Modoc and DPLL have an easier time with the satisfactory modifications, while model elimination has greater difficulty.

These tests, while incomplete, contain substantial evidence that autarky pruning overcomes the major inefficiency of model elimination. Research has improved the empirical performance of the basic DPLL algorithm substantially [Pre55, VGT96]. Whether further research can make Modoc competitive with the leading model-search methods remains to be seen.

Figure 16: Comparative performances on pigeon-hole formulas (unsat), and pigeon-hole formulas with one binary clause removed (sat). Operations reported: extensions for resolution methods and assignments (including unit and pure literals) for DPLL.
9 Conclusions and Future Work

We have introduced a method to incorporate autarky analysis into propositional resolution procedures. Preliminary results indicate that substantial gains of efficiency can be achieved. Future work should proceed along several directions, including an implementation with optimized data structures in an imperative language, such as C, research into more effective lemma techniques, and an extension to first-order theorem proving. Another open question is the worst-case complexity of Modoc or an improved version of Modoc. An efficient implementation needs to be evaluated empirically on a variety of applications, and compared with other high-performance satisfiability methods.

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References


