On the Invariance of Measured Equation of Invariance

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ABSTRACT

The key to the method of measured equation of invariance (MEI) is the postulate: "the MEI is independent of the excitation". The word "postulate" certainly just means a guess without rigorous proof. In the report, we theoretically prove that the MEI is independent of the excitation on the restriction of $O(h^2)$, here h is the discretization step. The error between the accurate solution and the solution of the MEI is shown. Besides, we show that the "metrons" are indeed not basis functions with substantial illustrations.

Keywords: Measured Equation of Invariance (MEI), postulate, error bound

1 Introduction

Measured Equation of Invariance(MEI) is a new concept in computational electromagnetics[MPCL92] [HLM94] [HM94] [HML94]. MEI is used to derive the local finite difference (FD) like equation at mesh boundaries where the conventional FD approach fails. It is demonstrated that the MEI technique can be used to terminate the meshes very close to the object boundary and still strictly preserves the sparsity of the FD equations. Therefore, the final system matrix encountered by MEI is a sparse matrix with size similar to that of integral equation methods. Therefore, the method of MEI definitely results in dramatic savings in computing time and memory usage compared to other known methods. It has been successfully used to analyze electromagnetic scattering problems [HLM94] [HM94], microwave integrated circuits, and IC interconnect parasitic extraction [HSD96] [SHD95].

The typical treatment of a FD procedure is to make meshes on and around the object in interest. By using loop integral method, the local FD equations can be easily deduced. But the derived FD equation is only applicable at interior nodes of the mesh. In paper [MPCL92], Mei postulated that the finite difference/element equations at the mesh boundary points may also be represented by a local linear equation. In conventional MEI, the distribution functions, called "metrons", are excited on the conductors or the surface of a penetrable media and the potential values on the MEI nodes are obtained from the integrals of the metrons multiplied by Green's function. Substituting the potential values at MEI nodes into MEI will lead to a system of linear algebraic equations with respect to the MEI coefficients, where each equation corresponding to one metron. The MEI or MEI coefficients are determined by solving the system of linear algebraic equations. Finally, the potential values at all nodes can be obtained by solving the system of linear algebraic equations which consist of FD equations at interior nodes and MEI at truncated mesh boundary nodes. The coefficient matrix of the system of linear algebraic equations is a sparse matrix since in 2D cases, each row contains either five non-zero elements from FD equations or M(or less) non-zero elements from MEI. Here, M is at most three without considering diagonal nodes. It results in great savings in memory needs compared with BEM or MoM etc. Furthermore, the computing time is proportional to N^2 for solving a sparse matrix equation but N^3 for solving a full matrix equation. The order of coefficient matrix in MEI approach is much less than that in conventional FD methods with absorbing boundary conditions, because MEI can terminate the mesh very close to the region in which we are interested. These properties make the method of MEI a powerful tool for computational electromagnetics.

Nowadays however, some papers[JL94] [JL95] propose some doubts on the third postulation of the MEI coefficients: invariant to excitations, which is the main topic of this report. In addition, even they have some doubts on MEI, they still admit in the papers that MEI is an efficient technique for the truncation of mesh boundaries.

(b) 2D mesh of cross-section

Figure 1: A general cylinder and its 2D mesh scheme

2 Basic MEI idea

Considering the EM scattering problem of a general cylinder (not necessary conducting, may be penetrable media) as shown in Fig.1(a), and several layers of 2D mesh around the cross-section of the cylinder shown in Fig.1(b). For the sake of convenience and without loss of generality, let the horizontal and vertical discretization step size the same and denoted as h.

At the interior nodes of the mesh, the following 5-points finite difference equation (FD)

$$\sum_{i=0}^{4} c_i \phi(\bar{r}_i) = O(h^4) \tag{1}$$

can be applied, where in Cartesian coordinate system and under uniform media assumption, $c_1 = c_2 = c_3 = c_4 = 1, c_0 = (kh)^2 - 4$, and $\bar{r}_i$ is the position vector of the ith node.

On the boundary nodes as shown in Fig.1(b), a different type of relation has to be applied, such as traditional E.W., M.W., and ABC's. Mei [MPCL92] postulated the existence of the following linear equation for boundary nodes

$$\sum_{i=0}^{3} c_i \phi(\bar{r}_i) = 0 \tag{2}$$

which is called MEI (measured equation of invariance), and $c_i, i = 0, 1, 2, 3$ are called MEI coefficients, which MEI postulated have three properties (i) location dependent, (ii) geometric specific, (iii) invariant to the excitation on the surface of the object. Among them, the third one, invariance to excitation is the base of the method of MEI. We are going to give a rigorous theoretical proof on this postulate.

3 Proof of Invariance to Excitation

The proof begins with definitions of some concepts.

Definition 1. Let C denote the continuous function space consisting of the continuous functions defined on the boundary Γ of the cylinder.

For any incident field, the induced current distribution $J(l)$ (l is the length along the boundary Γ) is always a continuous function, so $J(l) \in C$.

The scattered field $\phi(\bar{r})$ produced by the induced current $J(l)$ can be calculated by the following formula

$$\phi(\bar{r}) = \int_{\Gamma'} J(l')K(\bar{r}, \bar{r}')dl' = \pounds[J(l')] \tag{3}$$

where $K(\bar{r}, \bar{r}')$ is a kernel function, and $\pounds$ is the integral operator.

The field function $\phi(\bar{r})$ should satisfy the Helmholtz equation, so its derivatives of the second order should be continuous. According to the property of the kernel function $K(\bar{r}, \bar{r}')$, in fact, the nth ($n > 2$) order derivatives of the field function $\phi(\bar{r})$ are continuous.

Defining vector $\bar{\phi} = (\phi(\bar{r}_1), \phi(\bar{r}_2), \phi(\bar{r}_3), \phi(\bar{r}_0))$, here $\bar{r}_i$ is the position vector of the ith node of MEI, then

Definition 2. $\Phi \doteq \{\bar{\phi}, \phi(\bar{r}) = \pounds[J(l)], \forall J(l) \in C\}$ is the vector space consisting of vectors $\bar{\phi}$ produced by all currents.

Definition 3. $M \doteq \{$ $\bar{c} = (c_1, c_2, c_3, 1), c_i$ are any bounded complex numbers$\}$ is defined as the space of MEI coefficients vectors, here the MEI coefficient c_0 has been normalized to 1.

Definition 4. If $\bar{c} \cdot \bar{\phi} = \sum_{i=0}^{3} c_i \phi(\bar{r}_i) = 0$, then we say $\bar{c}$ is perpendicular to $\bar{\phi}$, or $\bar{c} - \bar{\phi}$. If $\forall \bar{\phi} \in \Phi, \bar{c} - \bar{\phi}$, then we say $\bar{c}$ is perpendicular to the space Φ, or $\bar{c} - \Phi$.

If $\bar{c} - \Phi$, we say $\bar{c}$ is independent of Φ. Since $\bar{c} \cdot \bar{\phi} = 0, \forall \bar{\phi} \in \Phi \Longleftrightarrow \bar{c} - \Phi$, so if we want to prove $\bar{c}$ is independent of space Φ, we only need to prove that $\forall \bar{\phi} \in \Phi, \bar{c} \cdot \bar{\phi} = 0$.

However, in numerical analysis, we usually have $|\bar{c} \cdot \bar{\phi}| \leq \varepsilon \neq 0, \forall \bar{\phi} \in \Phi$, here ε is a very small quantity. In this case obviously, $\bar{c}$ is not independent of space Φ, but we can say $\bar{c}$ is independent of space Φ on the sense of ε.

Let $p = \bar{c} \cdot \bar{\phi}$ as the projection from $\bar{c}$ to $\bar{\phi}$, then p describes the interrelation between $\bar{c}$ and $\bar{\phi}$. The less the projection p, the weaker the interrelation between $\bar{c}$ and $\bar{\phi}$.

We can rewrite the MEI Eq.2 as

$$\bar{c} \cdot \bar{\phi} = 0, \quad \mathit{or} \;\; \bar{c} - \bar{\phi}, \forall \bar{\phi} \in \Phi \tag{4}$$

which means $\bar{c}$ is rigorously independent of the equivalent current distribution on the surface of the scattering cylinder, or independent of incident field because the current distribution is actually induced by the incident field.

As mentioned above, the invariance to excitation is only a postulation or guess. Is there really a MEI coefficients vector $\bar{c}$ that is rigorously independent of the excitationΓ The question is answered by the following theorem.

Theorem Assume a vector $\bar{c}^*$ is perpendicular to three linear independent vectors of the space Φ, i.e.

$$\bar{c}^* \cdot \bar{\phi}_m = \sum_{i=0}^{3} c_i^* \phi_m(\bar{r}_i) = 0, \quad m = 1, 2, 3 \quad \phi_m \in \Phi \tag{5}$$

then,

$$\bar{c}^* \cdot \bar{\phi} = \sum_{i=0}^{3} c_i^* \phi(\bar{r}_i) = O(h^2) \quad \forall \phi \in \Phi \tag{6}$$

which means the MEI coefficient vector $\bar{c}^*$ is independent of excitation on the sense of $O(h^2)$.

Proof: $\forall \bar{\phi} \in \Phi$, define the projection from $\bar{c}^*$ to $\bar{\phi}$ as

$$p = \bar{c}^* \cdot \bar{\phi} = \sum_{i=0}^{3} c_i^* \phi(\bar{r}_i) \tag{7}$$

The condition Eq.5 is just a system of linear algebraic equation with respect to the MEI coefficients c_1^*, c_2^*, and c_3^*

$$\begin{bmatrix} \phi_1(\bar{r}_1) & \phi_1(\bar{r}_2) & \phi_1(\bar{r}_3) \\ \phi_2(\bar{r}_1) & \phi_2(\bar{r}_2) & \phi_2(\bar{r}_3) \\ \phi_3(\bar{r}_1) & \phi_3(\bar{r}_2) & \phi_3(\bar{r}_3) \end{bmatrix} \begin{bmatrix} c_1^* \\ c_2^* \\ c_3^* \end{bmatrix} = - \begin{bmatrix} \phi_1(\bar{r}_0) \\ \phi_2(\bar{r}_0) \\ \phi_3(\bar{r}_0) \end{bmatrix} \tag{8}$$

whose solution can be easily expressed according to Gramm's rule

$$c_i^* = \frac{D_i}{D} \qquad i = 1, 2, 3 \tag{9}$$

where

$$D = \begin{vmatrix} \phi_1(\bar{r}_1) & \phi_1(\bar{r}_2) & \phi_1(\bar{r}_3) \\ \phi_2(\bar{r}_1) & \phi_2(\bar{r}_2) & \phi_2(\bar{r}_3) \\ \phi_3(\bar{r}_1) & \phi_3(\bar{r}_2) & \phi_3(\bar{r}_3) \end{vmatrix} \tag{10}$$

$$D_1 = - \begin{vmatrix} \phi_1(\bar{r}_0) & \phi_1(\bar{r}_2) & \phi_1(\bar{r}_3) \\ \phi_2(\bar{r}_0) & \phi_2(\bar{r}_2) & \phi_2(\bar{r}_3) \\ \phi_3(\bar{r}_0) & \phi_3(\bar{r}_2) & \phi_3(\bar{r}_3) \end{vmatrix} \tag{11}$$

$$D_2 = - \begin{vmatrix} \phi_1(\bar{r}_1) & \phi_1(\bar{r}_0) & \phi_1(\bar{r}_3) \\ \phi_2(\bar{r}_1) & \phi_2(\bar{r}_0) & \phi_2(\bar{r}_3) \\ \phi_3(\bar{r}_1) & \phi_3(\bar{r}_0) & \phi_3(\bar{r}_3) \end{vmatrix} \tag{12}$$

$$D_3 = -\begin{vmatrix} \phi_1(\bar{r}_1) & \phi_1(\bar{r}_2) & \phi_1(\bar{r}_0) \\ \phi_2(\bar{r}_1) & \phi_2(\bar{r}_2) & \phi_2(\bar{r}_0) \\ \phi_3(\bar{r}_1) & \phi_3(\bar{r}_2) & \phi_3(\bar{r}_0) \end{vmatrix} \tag{13}$$

Therefore, the projection p can be expressed as

$$\begin{aligned} p &= \phi(\bar{r}_1)c_1^* + \phi(\bar{r}_2)c_2^* + \phi(\bar{r}_3)c_3^* + \phi(\bar{r}_0) \\ &= \frac{\phi(\bar{r}_1)D_1 + \phi(\bar{r}_2)D_2 + \phi(\bar{r}_3)D_3 + \phi(\bar{r}_0)D}{D} \\ &= \frac{D_4}{D} \end{aligned} \tag{14}$$

where

$$\begin{aligned} D_4 &= -\begin{vmatrix} \phi(\bar{r}_1) & \phi(\bar{r}_2) & \phi(\bar{r}_3) & \phi(\bar{r}_0) \\ \phi_1(\bar{r}_1) & \phi_1(\bar{r}_2) & \phi_1(\bar{r}_3) & \phi_1(\bar{r}_0) \\ \phi_2(\bar{r}_1) & \phi_2(\bar{r}_2) & \phi_2(\bar{r}_3) & \phi_2(\bar{r}_0) \\ \phi_3(\bar{r}_1) & \phi_3(\bar{r}_2) & \phi_3(\bar{r}_3) & \phi_3(\bar{r}_0) \end{vmatrix} \\ &= -\begin{vmatrix} \phi(\bar{r}_0) - h\phi'_s(\bar{r}_0) + 0.5h^2\phi''_s(\bar{r}_0) + O(h^3) & \phi(\bar{r}_0) - h\phi'_n(\bar{r}_0) + O(h^2) & \phi(\bar{r}_0) + h\phi'_s(\bar{r}_0) + 0.5h^2\phi''_s(\bar{r}_0) + O(h^3) & \phi(\bar{r}_0) \\ \phi_1(\bar{r}_0) - h\phi'_{s1}(\bar{r}_0) + 0.5h^2\phi''_{s1}(\bar{r}_0) + O(h^3) & \phi_1(\bar{r}_0) - h\phi'_{n1}(\bar{r}_0) + O(h^2) & \phi_1(\bar{r}) + h\phi'_{s1}(\bar{r}_0) + 0.5h^2\phi''_{s1}(\bar{r}_0) + O(h^3) & \phi_1(\bar{r}_0) \\ \phi_2(\bar{r}_0) - h\phi'_{s2}(\bar{r}_0) + 0.5h^2\phi''_{s2}(\bar{r}_0) + O(h^3) & \phi_2(\bar{r}_0) - h\phi'_{n2}(\bar{r}_0) + O(h^2) & \phi_2(\bar{r}) + h\phi'_{s2}(\bar{r}_0) + 0.5h^2\phi''_{s2}(\bar{r}_0) + O(h^3) & \phi_2(\bar{r}_0) \\ \phi_3(\bar{r}_0) - h\phi'_{s3}(\bar{r}_0) + 0.5h^2\phi''_{s3}(\bar{r}_0) + O(h^3) & \phi_3(\bar{r}_0) - h\phi'_{n3}(\bar{r}_0) + O(h^2) & \phi_3(\bar{r}) + h\phi'_{s3}(\bar{r}_0) + 0.5h^2\phi''_{s3}(\bar{r}_0) + O(h^3) & \phi_3(\bar{r}_0) \end{vmatrix} \\ &= -\begin{vmatrix} -h\phi'_s(\bar{r}_0) + 0.5h^2\phi''_s(\bar{r}_0) + O(h^3) & -h\phi'_n(\bar{r}_0) + O(h^2) & h\phi'_s(\bar{r}_0) + 0.5h^2\phi''_s(\bar{r}_0) + O(h^3) & \phi(\bar{r}_0) \\ -h\phi'_{s1}(\bar{r}_0) + 0.5h^2\phi''_{s1}(\bar{r}_0) + O(h^3) & -h\phi'_{n1}(\bar{r}_0) + O(h^2) & h\phi'_{s1}(\bar{r}_0) + 0.5h^2\phi''_{s1}(\bar{r}_0) + O(h^3) & \phi_1(\bar{r}_0) \\ -h\phi'_{s2}(\bar{r}_0) + 0.5h^2\phi''_{s2}(\bar{r}_0) + O(h^3) & -h\phi'_{n2}(\bar{r}_0) + O(h^2) & h\phi'_{s2}(\bar{r}_0) + 0.5h^2\phi''_{s2}(\bar{r}_0) + O(h^3) & \phi_2(\bar{r}_0) \\ -h\phi'_{s3}(\bar{r}_0) + 0.5h^2\phi''_{s3}(\bar{r}_0) + O(h^3) & -h\phi'_{n3}(\bar{r}_0) + O(h^2) & h\phi'_{s3}(\bar{r}_0) + 0.5h^2\phi''_{s3}(\bar{r}_0) + O(h^3) & \phi_3(\bar{r}_0) \end{vmatrix} \\ &= h^4\begin{vmatrix} \phi'_s(\bar{r}_0) & \phi'_n(\bar{r}_0) & \phi''_s(\bar{r}_0) & \phi(\bar{r}_0) \\ \phi'_{s1}(\bar{r}_0) & \phi'_{n1}(\bar{r}_0) & \phi''_{s1}(\bar{r}_0) & \phi_1(\bar{r}_0) \\ \phi'_{s2}(\bar{r}_0) & \phi'_{n2}(\bar{r}_0) & \phi''_{s2}(\bar{r}_0) & \phi_2(\bar{r}_0) \\ \phi'_{s3}(\bar{r}_0) & \phi'_{n3}(\bar{r}_0) & \phi''_{s3}(\bar{r}_0) & \phi_3(\bar{r}_0) \end{vmatrix} + O(h^5) \end{aligned} \tag{15}$$

where $\phi'_{si}(\bar{r}_0) = \frac{\partial}{\partial s}\phi_i(\bar{r}_0)$, $\phi'_{ni}(\bar{r}_0) = \frac{\partial}{\partial n}\phi_i(\bar{r}_0)$, and s and n are the two orthogonal directions the mesh is built on. Similarly,

$$D = -\begin{vmatrix} \phi_1(\bar{r}_1) & \phi_1(\bar{r}_2) & \phi_1(\bar{r}_3) \\ \phi_2(\bar{r}_1) & \phi_2(\bar{r}_2) & \phi_2(\bar{r}_3) \\ \phi_3(\bar{r}_1) & \phi_3(\bar{r}_2) & \phi_3(\bar{r}_3) \end{vmatrix}$$

$$
\begin{aligned}
&= -\begin{vmatrix} \phi_1(\bar{r}_0) - h\phi'_{s1}(\bar{r}_0) + O(h^2) & \phi_1(\bar{r}_0) - h\phi'_{n1}(\bar{r}_0) + O(h^2) & \phi_1(\bar{r}_0) + h\phi'_{s1}(\bar{r}_0) + O(h^2) \\ \phi_2(\bar{r}_0) - h\phi'_{s2}(\bar{r}_0) + O(h^2) & \phi_2(\bar{r}_0) - h\phi'_{n2}(\bar{r}_0) + O(h^2) & \phi_2(\bar{r}_0) + h\phi'_{s2}(\bar{r}_0) + O(h^2) \\ \phi_3(\bar{r}_0) - h\phi'_{s3}(\bar{r}_0) + O(h^2) & \phi_3(\bar{r}_0) - h\phi'_{n3}(\bar{r}_0) + O(h^2) & \phi_3(\bar{r}_0) + h\phi'_{s3}(\bar{r}_0) + O(h^2) \end{vmatrix} \\
&= 2h^2 \begin{vmatrix} \phi'_{s1}(\bar{r}_0) & \phi'_{n1}(\bar{r}_0) & \phi_1(\bar{r}_0) \\ \phi'_{s2}(\bar{r}_0) & \phi'_{n2}(\bar{r}_0) & \phi_2(\bar{r}_0) \\ \phi'_{s3}(\bar{r}_0) & \phi'_{n3}(\bar{r}_0) & \phi_3(\bar{r}_0) \end{vmatrix} + O(h^3) \qquad (16)
\end{aligned}
$$

Therefore, the projection

$$
p = \frac{h^2}{2} \frac{\begin{vmatrix} \phi'_s(\bar{r}_0) & \phi'_n(\bar{r}_0) & \phi''_s(\bar{r}_0) & \phi(\bar{r}_0) \\ \phi'_{s1}(\bar{r}_0) & \phi'_{n1}(\bar{r}_0) & \phi''_{s1}(\bar{r}_0) & \phi_1(\bar{r}_0) \\ \phi'_{s2}(\bar{r}_0) & \phi'_{n2}(\bar{r}_0) & \phi''_{s2}(\bar{r}_0) & \phi_2(\bar{r}_0) \\ \phi'_{s3}(\bar{r}_0) & \phi'_{n3}(\bar{r}_0) & \phi''_{s3}(\bar{r}_0) & \phi_3(\bar{r}_0) \end{vmatrix}}{\begin{vmatrix} \phi'_{s1}(\bar{r}_0) & \phi'_{n1}(\bar{r}_0) & \phi_1(\bar{r}_0) \\ \phi'_{s2}(\bar{r}_0) & \phi'_{n2}(\bar{r}_0) & \phi_2(\bar{r}_0) \\ \phi'_{s3}(\bar{r}_0) & \phi'_{n3}(\bar{r}_0) & \phi_3(\bar{r}_0) \end{vmatrix}} = O(h^2) \qquad (17)
$$

The theorem is proved.

It should be noted that in the proof, if $\bar{\phi} \in \Phi$, but $\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3 \notin \Phi$, the conclusion is still right, which means the MEI with $O(h^2)$ residue is not unique, or there are infinite sets of MEI coefficients that are independent of the excitation on the sense of $O(h^2)$.

4 Conclusion

In this report, we rigorously prove that in MEI, the residue of the projection from MEI coefficients to any field distribution produced by possible sources on the surfaces of the interesting object is in the order of $O(h^2)$, where h is the discretization error, which verify the third key postulation of MEI and gave an error bound.

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