Duality between L-bases and B-bases

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ABSTRACT

L-bases and B-bases are two important classes of polynomial bases used for representing surfaces in CAGD. The Bézier and multinomial bases are special cases of both L-bases and B-bases. We establish that certain proper subclasses of bivariate Lagrange and Newton bases are L-bases and certain proper subclasses of power and Newton dual bases are B-bases. A geometric point-line duality between B-bases and L-bases is described and used to investigate the duality between geometric representations for bivariate Bézier and multinomial bases, Lagrange and power bases, and Newton and Newton dual bases for surfaces. Under this geometric duality, lines in L-bases correspond to points or vectors in B-bases and concurrent lines map to collinear points and vice-versa. The generalized de Boor-Fix formula for surfaces also provides an algebraic duality between L-bases and B-bases. This algebraic duality between B-bases and L-bases can be used to develop change of basis algorithms between any two of these bases. We describe, in particular, a change of basis algorithm from a bivariate Lagrange L-basis to a bivariate Bézier basis.

Keywords: algorithms, B-bases, change of basis, CAGD, duality, L-bases, surfaces.

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1. Introduction

Lagrange and Newton bases for surfaces are very useful for interpolating point and derivative data. Here we shall establish that certain proper subclasses of these bivariate bases arise as L-bases - that is, bases that can be factored in a special way into products of linear polynomials. Other important examples of bivariate L-bases include the Bézier and multinomial bases.

We shall also introduce the power and Newton dual bases for surfaces and establish that certain subclasses of these bases arise as B-bases - that is, bases that are blending functions for B-patches. B-patches were first introduced by Seidel [Sei91] and later shown to agree with multivariate B-splines on a certain region of the parameter domain [DMS92]. The basis functions for the B-patches, that is the B-bases, are known to be local multivariate generalizations of univariate B-splines. Other important examples of bivariate B-bases again include the Bézier and multinomial bases.

A duality principle relating homogeneous L-bases and B-bases will be used to show that the Lagrange and generalized Newton bases are dual to the power and generalized Newton dual bases. This duality between homogeneous L-bases and B-bases can be derived using a multivariate polynomial identity [CM92] or by generalizing the de Boor-Fix dual functionals from curves [dBF73] to surfaces [LG94d]. We demonstrate that this algebraic principle of duality between homogeneous L-bases and B-bases gives rise to a geometric principle of duality between geometric representations for affine L-bases and B-bases. Under this geometric principle of duality, lines representing L-bases correspond to points or vectors representing B-bases and concurrent lines map to collinear points and vice-versa. This interpretation unifies a wide variety of bivariate polynomial bases including Lagrange, Newton, power, Bézier, multinomial, and Newton dual bases. This unification yields an elegant change of basis algorithm between any two of these bases with computational complexity $O(n^3)$. In particular, we shall present the change of basis algorithm between the Lagrange and Bézier bases.

Our work easily generalizes to higher dimensions. Nevertheless, for the sake of simplicity, the results are presented and derived here only for surfaces.

This paper is organized in the following manner. Section 2 reviews the definitions of L-bases and B-bases. Section 3 focuses on duality: A geometric point-line duality is introduced between representations for B-bases and L-bases, and an algebraic duality is formulated from a generalization of the de Boor-Fix formula from curves to surfaces. Many interesting examples of dual bases are provided in Section 4 including the general bivariate Bézier and multinomial bases, special Lagrange and power bases, and certain Newton and Newton dual bases. In Section 5 we turn our attention to algebraic duality. Here we mention various dual formulas and algorithms based on the algebraic duality between B-bases and L-bases that arises from the generalized de Boor-Fix formula. We focus, in particular, on change of basis algorithms for L-bases, and we exhibit these procedures by converting a bivariate polynomial from Lagrange to Bézier form. We conclude in Section 6 with a short summary of our work and a brief discussion of future research.

Throughout this paper, we shall adopt the following notation. A multi-index α is a 3-tuple of non-negative integers. If $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, then $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and $\alpha! = \alpha_1!\alpha_2!\alpha_3!$. Other multi-indices will be denoted by β and γ. A unit multi-index e_k is a

3-tuple with 1 in the k-th position and 0 everywhere else. Scalar indices will be denoted by i, j, k, l. Finally, given a homogeneous polynomial $f(x,y,z)$, $D^\alpha f$ denotes $\frac{\partial^{|\alpha|} f}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}}$.

2. Bases

Here we review the basic definitions and certain well-known properties of homogeneous and affine L-bases and B-bases. We also provide geometric interpretations for the algebraic entities associated with these bases.

2.1 L-bases

A collection $\mathcal{L}$ of 3 sets $\{L_{1,j}\}$, $\{L_{2,j}\}$, $\{L_{3,j}\}$, $j = 1, \cdots, n$ of linear homogeneous (resp. affine) polynomials in three (resp. two) variables is called a *knot-net* of homogeneous (resp. affine) polynomials if $(L_{1,\alpha_1+1}, L_{2,\alpha_2+1}, L_{3,\alpha_3+1})$ are linearly (resp. affinely) independent polynomials for $0 \leq |\alpha| \leq n-1$. A homogeneous (resp. affine) L-basis $\{l_\alpha^n, |\alpha| = n\}$ is a collection of $\binom{n+2}{2}$ trivariate (resp. bivariate) polynomials defined as follows:

$$l_\alpha^n = \prod_{i=1}^{\alpha_1} L_{1i} \prod_{j=1}^{\alpha_2} L_{2j} \prod_{k=1}^{\alpha_3} L_{3k}. \tag{2.1}$$

It is well-known that $\{l_\alpha^n, |\alpha| = n\}$ is, in fact, a homogeneous (resp. affine) basis for the space of homogeneous (resp. affine) polynomials of degree n on R^3 (resp. R^2) [CM92].

By associating the homogeneous polynomial $L = ax + by + cz$ with the affine polynomial $A = ax + by + c$, one can define a one-to-one correspondence between the knot-net of homogeneous and affine polynomials and between the homogeneous and affine L-bases. Due to this one-to-one correspondence between homogeneous and affine L-bases, in the following discussions we shall refer to either the homogeneous or affine L-basis, whichever is more convenient or intuitive in the particular context.

Furthermore, we assign to each homogeneous (resp. affine) polynomial, the following geometric interpretation. The polynomial $ax + by + cz$ (resp. $ax + by + c$) corresponds to the line in the projective (resp. affine) plane defined by the equation $ax + by + cz = 0$ (resp. $ax + by + c = 0$). In particular, the polynomial cz (resp. c) corresponds to the line at infinity in the projective plane. Observe that this correspondence between the lines and polynomials depends on the coordinate system and is unique only up to constant multiples. Nevertheless, we shall identify the polynomial with the line and vice-versa in the following discussions, whenever the coordinate system and constant multiples are irrelevant for the context at hand. The advantage of this correspondence is to allow us to think of algebraic entities such as polynomials in terms of geometric entities such as lines.

2.2 B-bases

A collection $\mathcal{U}$ of 3 sets $\{\mathbf{u}_{1,j}\}$, $\{\mathbf{u}_{2,j}\}$, $\{\mathbf{u}_{3,j}\}$, $j = 1, \cdots, n$ of vectors in R^3 is called a *knot-net* of vectors if $(\mathbf{u}_{1,\alpha_1+1}, \mathbf{u}_{2,\alpha_2+1}, \mathbf{u}_{3,\alpha_3+1})$ are linearly independent vectors in R^3 for $0 \leq |\alpha| \leq n-1$. One can write any vector $\mathbf{u}$ in R^3 in terms of the basis $(\mathbf{u}_{1,\alpha_1+1}, \mathbf{u}_{2,\alpha_2+1}, \mathbf{u}_{3,\alpha_3+1})$ so that

$$\mathbf{u} = \sum_{k=1}^{3} h_{k,\alpha}(\mathbf{u}) \mathbf{u}_{k,\alpha_k+1}.$$

Notice that $h_{k,\alpha}$ are trivariate homogeneous polynomials.

A homogeneous B-patch of degree n over the knot-net $\mathcal{U}$ is a trivariate homogeneous polynomial $B : R^3 \rightarrow R^m$ defined by the following recurrence. The initial conditions for the recurrence are given by setting $C_\alpha^0(\mathbf{u}) = C_\alpha \, \epsilon R^m$ for $|\alpha| = n$. The recurrence is constructed for $|\alpha| = n - l$, $l = 1, \cdots, n$ by

$$C_\alpha^l(\mathbf{u}) = \sum_{k=1}^{3} h_{k,\alpha}(\mathbf{u}) C_{\alpha+e_k}^{l-1}(\mathbf{u}). \tag{2.2}$$

The homogeneous B-patch is then defined as $B(\mathbf{u}) = C_0^n(\mathbf{u})$. This algorithm is known as the *up* recurrence; it generalizes to surfaces the de Boor evaluation algorithm for B-spline curves [dB72]. A homogeneous B-basis $\{b_\alpha^n, |\alpha| = n\}$ is a collection of $\binom{n+2}{2}$ homogeneous trivariate polynomials from R^3 to R defined by choosing the constants $C_\beta \epsilon R$ as follows:

$$\begin{aligned} C_\beta &= 1 \;\; if \;\; \beta = \alpha \\ &= 0 \;\; otherwise. \end{aligned}$$

It has been shown that $\{b_\alpha^n, |\alpha| = n\}$ is, in fact, a basis for the space of homogeneous polynomials on R^3 [Sei91]. Moreover, an arbitrary homogeneous B-patch of degree n can be represented in terms of a homogeneous B-basis as follows:

$$B(\mathbf{u}) = \sum_{|\alpha|=n} C_\alpha b_\alpha^n(\mathbf{u}).$$

Associating a knot-net of points in R^2 to a knot-net of vectors in R^3 is more subtle than associating lines in the plane with the knot-nets of linear polynomials. First, with any point $\mathbf{v} = (a, b)$ in R^2, we associate the vector $\mathbf{u} = (a, b, 1)$ in R^3, and with any vector $\mathbf{v} = (a, b)$ in R^2, we associate the vector $\mathbf{u} = (a, b, 0)$ in R^3.

Now, we need to explore what we mean by the linear independence of points and vectors in R^2. Given any three points or vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ in R^2, there are three distinct cases to consider:

1. $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are all points. Three points in R^2 are said to be linearly independent iff they are not collinear or alternatively iff they form a non-degenerate triangle.
2. Two of the three, say $\mathbf{v}_1$ and $\mathbf{v}_2$, are points and the third one $\mathbf{v}_3$ is a vector. These entities are said to be linearly independent iff $\mathbf{v}_1 \neq \mathbf{v}_2$ and the vector $\mathbf{v}_3$ does not lie along the straight line determined by the two points $\mathbf{v}_1$ and $\mathbf{v}_2$.
3. Two of the three, say $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors and the third one $\mathbf{v}_3$ is a point. These entities are said to be linearly independent iff the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent in R^2.

The fourth and only remaining case when $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are all vectors is not of interest to us because three vectors in R^2 are always linearly dependent. It is easy to verify that with the correspondence between points and vectors of R^2 and vectors of R^3 defined above, three points or vectors in R^2 are linearly independent iff the corresponding vectors in R^3 are linearly independent.

There is an important alternative way of thinking about vectors in R^2. A vector (a, b) in R^2 can also be thought of as the point at infinity in the direction of the vector (a, b), that is, the point at infinity on the line: $bx - ay = 0$ in the projective plane. With this interpretation all three cases listed above can be combined into a single case where three points or vectors in R^2 are linearly independent iff they are not collinear in the projective plane. For this reason, instead of referring to $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ as points or vectors in R^2, we shall always refer to them as points with the understanding that a point includes a point at infinity, which can be thought of as a vector in R^2. The distinction between points and vectors in R^2 will be emphasized only when it is relevant to the context. It is remarkable that this distinction vanishes after homogenization and that homogenization holds the key to dealing with point and derivative information on equal footing.

Now formally, the above correspondence associates points in the projective plane with vectors in R^3, where a point (a, b) in the affine plane corresponds to the vector $(a, b, 1)$ in R^3, while a vector (a, b) in the affine plane or equivalently a point $(a, b, 0)$ in the projective plane corresponds to the vector $(a, b, 0)$ in R^3.

To describe the correspondence the other way around: to a vector $\mathbf{u} = (a, b, c)$ in R^3, we shall associate the point $\mathbf{v} = (\frac{a}{c}, \frac{b}{c})$ in the affine plane whenever $c \neq 0$ and the vector $\mathbf{v} = (a, b)$ in the affine plane whenever $c = 0$. Equivalently, we shall associate the point (a, b, c) in the projective plane to the vector (a, b, c) in R^3. With this correspondence, it is again easy to verify that three vectors in R^3 are linearly independent iff the corresponding points or vectors in the affine plane are linearly independent. Observe that this correspondence between the knot-net of vectors in R^3 and the knot-net of points in R^2 is not surjective, but when restricted to the subset of vectors in R^3 whose third component is either 1 or 0, it is one-to-one.

3. Duality between B-bases and L-bases

We now describe the duality between B-bases and L-bases from two different perspectives: algebraic and geometric.

3.1 de Boor-Fix Duality

Given a knot-net of vectors $\mathbf{u}_{i,j}$ in R^3, consider the knot-net of linear homogeneous polynomials $L_{i,j}$ defined by the correspondence:

$$(a, b, c) \leftrightarrow (ax + by + cz).$$

Let l_α^n be the L-basis functions defined by the knot-net $L_{i,j}$, and let b_β^n be the B-basis functions defined by the knot-net $\mathbf{u}_{i,j}$.

The bases l_α^n and b_β^n are related algebraically through the following bilinear form, also referred to as the bracket operator. Given any two homogeneous polynomials $f, g : R^3 \to R$ of degree n, define the bilinear form

$$[f, g](\mathbf{u}) = \frac{1}{n!} \sum_{|\alpha|=n} \frac{D^\alpha f(\mathbf{u}) * D^\alpha g(\mathbf{u})}{\alpha!}.$$

Note that this bracket operator depends on n, and therefore, strictly speaking, the notation $[f, g]_n$ is more appropriate. However, we shall suppress the subscript n, whenever it does not cause any ambiguity.

Theorem 1: *Generalized de Boor-Fix formula [LG94d]:* $[l_\alpha^n, b_\beta^n] = \delta_{\alpha\beta}$.

Corollary 1: *Cavaretta-Micchelli identity [CM92, LG95]:* $\sum_{|\alpha|=n} l_\alpha^n(x, y, z) b_\alpha^n(a, b, c) = (ax + by + cz)^n$.

Because of Theorem 1, the L-basis l_α^n can be used to represent the dual functionals for the B-basis b_β^n and vice-versa. We shall explore some of the consequences of this algebraic duality in Section 5.

3.2 Point-Line Duality

The correspondence $(a, b, c) \leftrightarrow ax + by + cz = (a, b, c) \cdot (x, y, z)$ associates to each vector in R^3 a homogeneous trivariate polynomial. Earlier we saw that vectors in R^3 correspond to points in the projective plane (or points and vectors in the affine plane), and homogeneous trivariate polynomials correspond to lines in the projective plane (or lines in the affine plane plus the line at infinity). Thus B-bases are represented by knot-nets of points $\mathbf{u}_{ij}$ in the projective plane and L-bases by knot-nets of lines L_{ij} in the projective plane. We say that a B-basis and an L-basis are dual bases if their knot-nets are related by the correspondence $L_{ij} = \mathbf{u}_{ij} \cdot (x, y, z)$. Under this correspondence points in the projective plane are mapped to lines in the projective plane and collinear points are mapped to concurrent lines.

Figure 3.1 summarizes the relationships between dual B-bases and L-bases, as well as the algebra and geometry underlying their associated knot-nets. A double arrow denotes a 1-1 correspondence; a solid arrow indicates that the correspondence is many to one; and a dotted 1-sided arrow means that the correspondence is not onto. Figure 3.1 can be

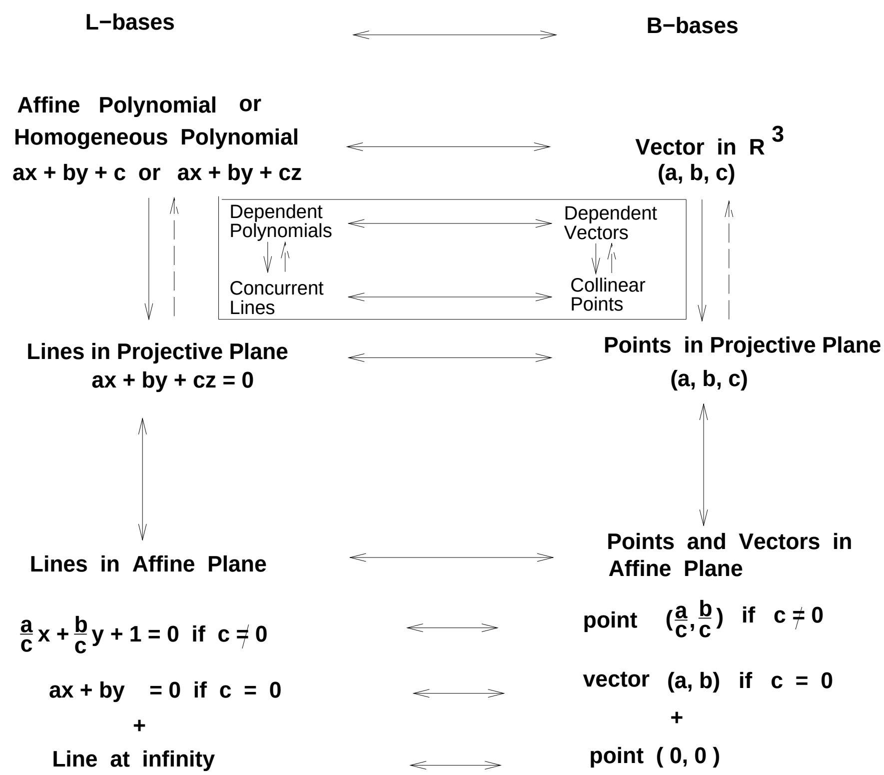

Figure 3.1: Point-line duality

made into a one-to-one correspondence between all the categories simply by restricting our attention to those homogeneous or affine polynomials, or those vectors in R^3, which have third component either 1 or 0. This restriction amounts to losing the additional flexibility of considering bases which are the same up to constant multiples.

4. Examples of Dual bases

In this section we discuss three sets of examples of dual B-bases and L-bases: dual Bézier and multinomial bases, dual Lagrange and power bases, and dual Newton and Newton dual bases. We begin by showing how each of these bases can be realized as a B-basis or an L-basis by constructing the appropriate knot-nets. We go on to discuss the geometry of these knot-nets as well as the geometry of the knot-nets for the corresponding dual bases. Later we shall see that while the correspondence at the homogenized level is simpler and more elegant algebraically, the point-line correspondence at the affine level provides better geometric insight.

4.1 Duality between Bézier and Multinomial Bases

This section explains how to realize any Bézier or multinomial basis as a special case of both B-bases and L-bases. We also introduce the hybrid BézierMultinomial (BM) basis in order to help investigate the duality between bivariate Bézier and multinomial bases. We shall refer to a B-basis as a *uniform* B-basis if the knot-net $\mathbf{u}_{ij}$ satisfies the property: $\mathbf{u}_{ij} = \mathbf{u}_i$ for $j = 1, \cdots, n$. A uniform L-basis is defined in an analogous manner.

4.1.1 Bézier Bases

First we describe how Bézier bases can be realized as special cases of B-bases. Let $\mathbf{u}_1 = (a_1, b_1, c_1)$, $\mathbf{u}_2 = (a_2, b_2, c_2)$ and $\mathbf{u}_3 = (a_3, b_3, c_3)$ be three linearly independent vectors in R^3 such that $c_i \neq 0$ for $i = 1, 2, 3$. Choose the uniform knot-net of vectors $\mathbf{u}_{i,j} = \mathbf{u}_i, 1 \leq j \leq n$. Then the corresponding B-basis is a homogeneous Bézier basis. For example, if $\mathbf{u}_1 = (1, 0, 1)$, $\mathbf{u}_2 = (0, 1, 1)$ and $\mathbf{u}_3 = (0, 0, 1)$, then it is easy to verify that the B-basis functions are also the homogeneous Bézier basis functions; that is,

$$b^n_\alpha(x, y, z) = \frac{n!}{\alpha!} x^{\alpha_1} y^{\alpha_2} (z - x - y)^{\alpha_3}.$$

More generally, if $\mathbf{u}_1 = (a_1, b_1, 1)$, $\mathbf{u}_2 = (a_2, b_2, 1)$ and $\mathbf{u}_3 = (a_3, b_3, 1)$, then it can readily be verified that the B-basis functions are indeed homogeneous Bézier basis functions; that is,

$$b^n_\alpha(x, y, z) = \frac{n!}{\alpha!} h_1^{\alpha_1} h_2^{\alpha_2} h_3^{\alpha_3} z^n,$$

where (h_1, h_2, h_3) are the barycentric coordinates of the point $(\frac{x}{z}, \frac{y}{z})$ with respect to the points (a_1, b_1), (a_2, b_2) and (a_3, b_3). Even more generally, if $c_i \neq 0$ for $i = 1, 2, 3$ and $\mathbf{u}_1 = (a_1, b_1, c_1)$, $\mathbf{u}_2 = (a_2, b_2, c_2)$ and $\mathbf{u}_3 = (a_3, b_3, c_3)$, then it can be verified that the B-basis functions are again homogeneous Bézier basis functions; this time,

$$b^n_\alpha(x, y, z) = \frac{n!}{\alpha!} (\frac{h_1}{c_1})^{\alpha_1} (\frac{h_2}{c_2})^{\alpha_2} (\frac{h_3}{c_3})^{\alpha_3} z^n,$$

where (h_1, h_2, h_3) are the barycentric coordinates of the point $(\frac{x}{z}, \frac{y}{z})$ with respect to the points $(\frac{a_1}{c_1}, \frac{b_1}{c_1})$, $(\frac{a_2}{c_2}, \frac{b_2}{c_2})$ and $(\frac{a_3}{c_3}, \frac{b_3}{c_3})$.

We can also realize Bézier bases as special cases of L-bases. Let $L_1 = a_1x + b_1y + c_1z$, $L_2 = a_2x + b_2y + c_2z$ and $L_3 = a_3x + b_3y + c_3z$ be three linearly independent polynomials. Furthermore, assume that the following three conditions are satisfied: $a_1b_2 - a_2b_1 \neq 0$, $a_2b_3 - a_3b_2 \neq 0$, and $a_3b_1 - a_1b_3 \neq 0$, that is, no two of the associated lines are parallel. Choose the uniform knot-net of polynomials $L_{i,j} = L_i, 1 \leq j \leq n$. Then the corresponding L-basis is a homogeneous Bézier basis up to constant multiples. Indeed, one can easily verify that up to constant multiples this L-basis is the homogenized Bézier basis defined by the three intersection points of L_1, L_2 and L_3: $\mathbf{u}_1 = (b_2c_3 - b_3c_2, a_3c_2 - a_2c_3, a_2b_3 - a_3b_2)$, $\mathbf{u}_2 = (b_3c_1 - b_1c_3, a_1c_3 - a_3c_1, a_3b_1 - a_1b_3)$ and $\mathbf{u}_3 = (b_1c_2 - b_2c_1, a_2c_1 - a_1c_2, a_1b_2 - a_2b_1)$. In fact with this choice of points, the linear L-basis, which is the same as the barycentric coordinates with respect to the triangle defined by these three points, is precisely $\frac{a_2b_3 - a_3b_2}{\Delta}L_1(\mathbf{u})$, $\frac{a_3b_1 - a_1b_3}{\Delta}L_2(\mathbf{u})$, and $\frac{a_1b_2 - a_2b_1}{\Delta}L_3(\mathbf{u})$, (or alternatively, $\frac{L_1(\mathbf{u})}{L_1(\mathbf{u}_1)}$, $\frac{L_2(\mathbf{u})}{L_2(\mathbf{u}_2)}$, and $\frac{L_3(\mathbf{u})}{L_3(\mathbf{u}_3)}$) where Δ is the determinant of the matrix defined by (a_1, b_1, c_1), (a_2, b_2, c_2) and (a_3, b_3, c_3). In particular, $L_1 = x$, $L_2 = y$ and $L_3 = -x - y + z$, yields the standard homogeneous Bézier basis, up to constant multiples, that is, $l_\alpha^n = x^{\alpha_1}y^{\alpha_2}(z - x - y)^{\alpha_3}$. In summary, given a triangle, we can use the vertices to define the Bézier basis – this is the B-basis point of view or we can use the lines to define the Bézier basis – this is the L-basis point of view.

4.1.2 Multinomial Bases

The multinomial basis is the standard generalization of the monomial basis to the multivariate setting. For example, the basis 1, x, y, x^2, xy and y^2 is the bivariate multinomial basis of degree 2. Sometimes the terminology Taylor basis or power basis is also used instead of monomial or multinomial basis. However, we shall refer to this basis as the multinomial basis in accordance with [GB92] and reserve the term power basis for the basis discussed later in Section 4.2.2. The standard multinomial basis is defined by the origin $(0, 0)$ and the unit vectors $(1, 0)$ and $(0, 1)$. The most general multinomial basis is similarly defined by a point and two linearly independent vectors and is discussed below.

We first describe how to realize multinomial bases as special cases of B-bases. Let $\mathbf{u}_1 = (a_1, b_1, c_1)$, $\mathbf{u}_2 = (a_2, b_2, c_2)$ and $\mathbf{u}_3 = (a_3, b_3, c_3)$ be three linearly independent vectors in R^3 such that $c_1 = c_2 = 0$. Observe that by the linear independence condition $c_3 \neq 0$. Choose the uniform knot-net of vectors $\mathbf{u}_{i,j} = \mathbf{u}_i, 1 \leq j \leq n$. Then the corresponding B-basis is a homogeneous multinomial basis up to constant multiples. In other words, the multinomial basis is defined by a point and two linearly independent vectors in R^2. The simplest and most popular example of this construction is obtained by setting $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (0, 1, 0)$ and $\mathbf{u}_3 = (0, 0, 1)$. In this case it is easy to verify that the B-basis functions are homogeneous multinomial basis functions, and that

$$b_\alpha^n(x, y, z) = \frac{n!}{\alpha!}x^{\alpha_1}y^{\alpha_2}z^{\alpha_3}.$$

If a homogeneous polynomial $B(\mathbf{u})$ has coefficients C_α with respect to the standard multinomial B-basis, then

$$B(\mathbf{u}) = \sum_{|\alpha|=n} \frac{n!}{\alpha!}C_\alpha x^{\alpha_1}y^{\alpha_2}z^{\alpha_3},$$

and the coefficients C_α represent, up to constant multiples, the directional derivatives of the polynomial $B(\mathbf{u})$ at the point $(0, 0)$ along the directions $(1, 0)$ and $(0, 1)$. The multinomial basis defined by a point $\mathbf{v}_1$ and two vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ is a generalization where the coefficients of a polynomial with respect to this multinomial basis represent, up to constant multiples, the directional derivatives of this polynomial at the point $\mathbf{v}_1$ along the directions $\mathbf{v}_2$ and $\mathbf{v}_3$. As an example, if $\mathbf{u}_1 = (1, -1, 0)$, $\mathbf{u}_2 = (1, 1, 0)$ and $\mathbf{u}_3 = (0, 0, 1)$, then it can readily be verified that the B-basis functions are again homogeneous multinomial basis functions; that is,

$$b_\alpha^n(x, y, z) = \frac{n!}{\alpha!}\left(\frac{x-y}{2}\right)^{\alpha_1}\left(\frac{x+y}{2}\right)^{\alpha_2} z^{\alpha_3},$$

where $\frac{x+y}{2}$ and $\frac{x-y}{2}$ represent the directions $(1, -1)$ and $(1, 1)$ along which the multinomial basis is formed instead of along the usual directions $(1, 0)$ and $(0, 1)$. As another example, if $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (0, 1, 0)$ and $\mathbf{u}_3 = (a_3, b_3, 1)$, then it can readily be verified that the B-basis functions are indeed homogeneous multinomial basis functions; this time,

$$b_\alpha^n(x, y, z) = \frac{n!}{\alpha!}(x - a_3 z)^{\alpha_1}(y - b_3 z)^{\alpha_2} z^{\alpha_3},$$

where the multinomial basis is formed at (a_3, b_3) along the usual directions $(1, 0)$ and $(0, 1)$. More generally, if $\mathbf{u}_1 = (a_1, b_1, 0)$, $\mathbf{u}_2 = (a_2, b_2, 0)$ and $\mathbf{u}_3 = (a_3, b_3, 1)$, then the homogeneous B-basis functions are

$$b_\alpha^n(x, y, z) = \frac{n!}{\alpha!}\left(\frac{b_2(x - a_3 z) - a_2(y - b_3 z)}{a_1 b_2 - a_2 b_1}\right)^{\alpha_1}\left(\frac{-b_1(x - a_3 z) + a_1(y - b_3 z)}{a_1 b_2 - a_2 b_1}\right)^{\alpha_2} z^{\alpha_3},$$

where the multinomial basis is formed at (a_3, b_3) along the directions (a_1, b_1) and (a_2, b_2).

We can also realize multinomial bases as special cases of L-bases. Let $L_1 = a_1 x + b_1 y + c_1 z$, $L_2 = a_2 x + b_2 y + c_2 z$ and $L_3 = z$ be three linearly independent polynomials. Observe that by the linear independence condition, it follows that $a_1 b_2 - a_2 b_1 \neq 0$; thus the lines corresponding to L_1 and L_2 are not parallel. Choose the uniform knot-net of polynomials $L_{i,j} = L_i, 1 \leq j \leq n$. Then one can easily verify that this L-basis is indeed the homogenized multinomial basis defined by the vectors $(\frac{b_2}{k}, -\frac{a_2}{k})$ and $(-\frac{b_1}{k}, \frac{a_1}{k})$ and the point $(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1})$, where $k = a_1 b_2 - a_2 b_1$. In particular choosing $L_1 = x$, $L_2 = y$ and $L_3 = z$, yields the standard homogeneous multinomial basis; that is $l_\alpha^n = x^{\alpha_1} y^{\alpha_2} z^{\alpha_3}$. Also choosing $L_1 = x - az$, $L_2 = y - bz$ and $L_3 = z$, yields the homogeneous multinomial basis defined by the point (a, b) and the unit vectors $(1, 0)$ and $(0, 1)$; that is $l_\alpha^n = (x - az)^{\alpha_1}(y - bz)^{\alpha_2} z^{\alpha_3}$.

4.1.3 Hybrid BézierMultinomial(BM) Bases

We now introduce hybrid BézierMultinomial (BM) bases in order to help describe the duality between Bézier and multinomial bases in the next section 4.1.4. A Bézier B-basis is defined by three points, while a multinomial B-basis is defined by a point and two vectors. A hybrid BézierMultinomial basis is defined by two points and a vector.

Any hybrid BM basis can be realized as a B-basis as follows: Let $\mathbf{u}_1 = (a_1, b_1, c_1)$, $\mathbf{u}_2 = (a_2, b_2, c_2)$ and $\mathbf{u}_3 = (a_3, b_3, c_3)$ be three linearly independent vectors in R^3 such that $c_1 = 0$, $c_2 \neq 0$ and $c_3 \neq 0$. Choose the uniform knot-net of vectors $\mathbf{u}_{i,j} = \mathbf{u}_i, 1 \leq j \leq n$. The corresponding B-basis will be referred to as a hybrid homogeneous BM basis. This basis is formed by choosing 2 points and a vector. For example, if $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (0, 0, 1)$ and $\mathbf{u}_3 = (0, 1, 1)$, then it can readily be verified that the B-basis functions are

$$b_\alpha^n(x, y, z) = \frac{n!}{\alpha!} x^{\alpha_1}(z - y)^{\alpha_2} y^{\alpha_3}.$$

If a homogeneous polynomial $B(\mathbf{u})$ has coefficients C_α with respect to this hybrid BM basis, that is,

$$B(\mathbf{u}) = \sum_{|\alpha|=n} C_\alpha \frac{n!}{\alpha!} x^{\alpha_1}(z-y)^{\alpha_2} y^{\alpha_3},$$

the coefficient C_{n00} represents, up to constant multiples, the directional derivative of $B(\mathbf{u})$ of order n in the direction of the vector $(1,0)$. The coefficients $C_{k,n-k,0}$ (resp. $C_{l,0,n-l}$) represent, up to constant multiples, the directional derivatives of $B(\mathbf{u})$ of order k in the direction of the vector $(1,0)$ evaluated at the point $(0,0)$ (resp. the directional derivatives of $B(\mathbf{u})$ of order l in the direction of the vector $(1,0)$ evaluated at the point $(0,1)$). This interpretation of the coefficients of a polynomial can be extended easily to the case when the polynomial is expressed in a general hybrid BM basis defined by 2 points and a vector.

We can also realize a hybrid BM basis as an L-basis. Let $L_1 = a_1x + b_1y + c_1z$, $L_2 = a_2x + b_2y + c_2z$ and $L_3 = a_3x + b_3y + c_3z$ be three linearly independent polynomials. Let us choose the knot-net of polynomials $L_{i,j} = L_i, 1 \le j \le n$. The restriction that $a_1b_2 - a_2b_1 \ne 0$, $a_1b_3 - a_3b_1 \ne 0$ and $a_2b_3 - a_3b_2 \ne 0$ defines a homogeneous Bézier basis. The restriction that $a_3 = b_3 = 0$ defines a multinomial basis. It is easy to verify that the only remaining restriction that maintains linear independence is $a_1b_2 - a_2b_1 \ne 0$, $a_2b_3 - b_2a_3 \ne 0$ and $a_1b_3 - a_3b_1 = 0$. Thus the lines corresponding to L_1 and L_3 are parallel. With this restriction the homogeneous L-basis is referred to as a hybrid BM basis. This hybrid basis is defined by the two points: $(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \frac{a_2c_1-a_1c_2}{a_1b_2-a_2b_1})$, $(\frac{b_2c_3-b_3c_2}{a_2b_3-a_3b_2}, \frac{a_3c_2-a_2c_3}{a_2b_3-a_3b_2})$ and the vector $(\frac{-b_1}{a_1b_2-a_2b_1}, \frac{a_1}{a_1b_2-a_2b_1}) = (\frac{-b_3}{a_3b_2-a_2b_3}, \frac{a_3}{a_3b_2-a_2b_3}) = (\frac{b_3c_1-b_1c_3}{\Delta}, \frac{a_1c_3-a_3c_1}{\Delta})$.

4.1.4 Duality

This section investigates the duality between bivariate Bézier and multinomial bases. First we describe the algebraic or de Boor-Fix duality between Bézier and multinomial bases. Then we shall comment upon the geometric duality between these bases.

A Bézier B-basis is defined by (a_1, b_1, c_1), (a_2, b_2, c_2) and (a_3, b_3, c_3) with $c_i \ne 0$ for $i = 1, 2, 3$. The dual L-basis is therefore defined by $L_1 = a_1x+b_1y+c_1z$, $L_2 = a_2x+b_2y+c_2z$, and $L_3 = a_3x + b_3y + c_3z$. Depending upon whether zero, one or two of the three terms $a_1b_2 - a_2b_1$, $a_2b_3 - a_3b_2$, and $a_3b_1 - a_1b_3$ are zero, the dual L-basis can be a Bézier basis, a hybrid BM basis, or a multinomial basis. More specifically, if all three terms are non-zero, then the dual L-basis is a Bézier basis; if exactly two of these terms are non-zero, then the dual L-basis is a hybrid BM basis, and finally if exactly one of these three terms is non-zero, then the dual L-basis is a multinomial basis. Note that these distinctions are very sensitive to the choice of the coordinate system. The upper diagram of Figure 4.1 presents three Bézier B-bases each defined by three points forming a right-angle triangle. The dual to these Bézier B-bases are shown immediately below them in the lower part of Figure 4.1. Depending upon the choice of the coordinate system, the dual bases are a multinomial basis, a BM basis, and a Bézier basis respectively.

The duality situation is similar for a multinomial B-basis defined by (a_1, b_1, c_1), (a_2, b_2, c_2) and (a_3, b_3, c_3) where exactly two of the three terms c_1, c_2 and c_3 are zero. Again the dual L-basis can be either a Bézier basis, a hybrid BM basis, or a multinomial basis depending upon how many of the three terms $a_1b_2 - a_2b_1$, $a_2b_3 - a_3b_2$, and $a_3b_1 - a_1b_3$ vanish.

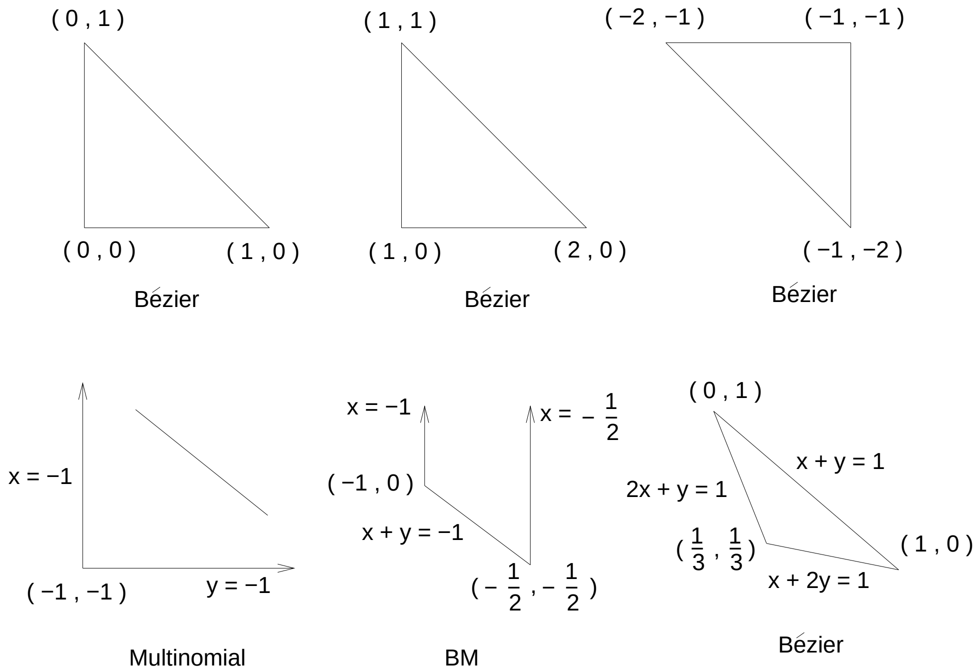

Figure 4.1: Duality between Bézier, multinomial and BM bases

In summary, a uniform B-basis – which can be either a Bézier basis, a hybrid BM basis, or a multinomial basis - is dual to a uniform L-basis – which can also be either a Bézier basis, a hybrid BM basis, or a multinomial basis.

These observations lead to the following geometric interpretation of duality between uniform B-bases and uniform L-bases. A Bézier B-basis is defined by three points; a hybrid BM B-basis by two points and a vector; a multinomial B-basis by a point and two vectors. Interpreting a vector as a point at infinity, a uniform B-basis is defined by three points. The dual L-basis is defined by three lines. Notice that the conditions $a_i b_j - a_j b_i = 0$ correspond to parallel lines in affine space and the number of parallel lines leads to the distinction between Bézier, BM, and multinomial L-bases. A Bézier L-basis is defined by three non-parallel lines in the affine plane. A BM L-basis is defined by three lines in the affine plane, exactly two of which are parallel. Finally, a multinomial L-basis is defined by the line at infinity and two non-parallel lines in the affine plane. In projective space where there are no parallel lines, these distinctions disappear.

Observe that it is *not* true that the three cases of uniform L-bases, namely Bézier basis, hybrid BM basis and multinomial basis, arise by taking i lines in the affine plane and $3-i$ lines at infinity for $i = 3, 2, 1$. In fact although there are many points at infinity, there is only one line at infinity. The multinomial L-basis arises by choosing exactly one line at infinity as described above. Alternatively, the three cases of uniform L-bases, namely Bézier basis, hybrid BM basis, and multinomial basis, arise by taking 3 lines such that i points of intersection of these lines lie in the affine plane and $3-i$ points of intersection lie at infinity for $i = 3, 2, 1$ respectively.

There is another potential source of confusion which is intriguing. Observe that the Bézier basis defined by the three points $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ is the same as, but *not* dual to, the L-basis defined by the three lines $\mathbf{v}_1\mathbf{v}_2$, $\mathbf{v}_2\mathbf{v}_3$, and $\mathbf{v}_3\mathbf{v}_1$. Such a duality, if it exists, should be referred to as *self-duality*. Under self-duality, the correspondence between vectors in R^3 and the homogeneous polynomials on R^3 would have to be defined from a set of 3-vectors to 3-polynomials and vice-versa rather than from a vector to a polynomial. In particular, a triple of vectors (a_1, b_1, c_1), (a_2, b_2, c_2) and (a_3, b_3, c_3) would correspond to the three homogeneous polynomials $(b_2c_3 - b_3c_2)x + (a_3c_2 - a_2c_3)y + (a_2b_3 - a_3b_2)z$, $(b_3c_1 - b_1c_3)x + (a_1c_3 - a_3c_1)y + (a_3b_1 - a_1b_3)z$, and $(b_1c_2 - b_2c_1)x + (a_2c_1 - a_1c_2)y + (a_1b_2 - a_2b_1)z$ under this self-dual correspondence. It would be very interesting to explore this self-duality. However, the duality presented in this work is not self-duality.

4.2 Duality between Lagrange and Power Bases

This section establishes that certain proper subclasses of bivariate Lagrange and power bases can be realized respectively as special cases of L-bases and B-bases and then investigates the duality between these special bases.

4.2.1 Lagrange Bases

Let $\{\ \{L_{ij}\},\ \{L_{2j}\},\ \{L_{3j}\},\ j = 1, \cdots, n\ \}$ be a knot-net of homogeneous polynomials. Suppose that the homogeneous polynomials $(L_{1,\alpha_1+1}, L_{2,\alpha_2+1}, L_{3,\alpha_3+1})$ are linearly dependent for $|\alpha| = n,\ 0 \leq \alpha_k \leq n-1$. The corresponding L-basis is then referred to as a Lagrange L-basis. We shall soon see that these dependency conditions give rise to a point-line configuration with $\binom{n+2}{2}$ points such that each of the $\binom{n+2}{2}$ L-basis functions vanishes at all the points except one, which justifies the terminology Lagrange L-basis.

To observe this, let us analyze the dependency conditions. Overloading the notation, let L_{ij} also denote the lines in the projective plane defined by the equations: $L_{ij} = 0$. The linear dependency condition on the polynomials L_{i,α_i+1} means that the projective lines L_{i,α_i+1} are concurrent for $|\alpha| = n,\ 0 \leq \alpha_k \leq n-1$. Let $\mathbf{v}_\alpha = \bigcap_{k=1}^{3} L_{k,\alpha_k+1}$ for $|\alpha| = n$, $0 \leq \alpha_k \leq n-1$. These intersections give rise to $\binom{n+2}{2} - 3$ points corresponding to $\binom{n+2}{2} - 3$ dependency conditions. To these points, we shall add three more points: $\mathbf{v}_{n00} = L_{31} \bigcap L_{21}$, $\mathbf{v}_{0n0} = L_{11} \bigcap L_{31}$, and $\mathbf{v}_{00n} = L_{11} \bigcap L_{21}$. It is easy to verify using Equation 2.1 that $l^n_\alpha(\mathbf{v}_\beta) = l^n_\alpha(\mathbf{v}_\alpha)\delta_{\alpha\beta}$. Therefore, $\{\frac{l^n_\alpha}{l^n_\alpha(\mathbf{v}_\alpha)}\}$ forms a Lagrange basis.

Now we are going to introduce certain interesting point-line configurations that give rise to bivariate Lagrange L-bases. To this extent, let us investigate the dependency conditions more closely in the affine plane. Let P_{ij} be the affine polynomials corresponding to the homogeneous polynomials L_{ij}. Overloading the notation, let P_{ij} also denote the lines in the affine plane defined by the equations: $P_{ij} = 0$. The linear dependency condition on the knot-net of polynomials corresponds to one of the following geometric conditions:

1. The lines (P_{1,α_1+1}, P_{2,α_2+1} and P_{3,α_3+1}) are distinct and concurrent; that is, they all pass through one common point $\mathbf{v}_\alpha = \bigcap_{k=1}^{3} P_{k,\alpha_k+1}$ when $|\alpha| = n$.
2. The lines (P_{1,α_1+1}, P_{2,α_2+1} and P_{3,α_3+1}) are distinct and parallel. Then $\{\ L_{1,\alpha_1+1}$, L_{2,α_2+1}, $L_{1,\alpha_1+1}\ \}$ all pass through a common point at infinity. For example, if $L_{1,\alpha_1+1} = k_1ax + k_1by + c_1z$, $L_{2,\alpha_2+1} = k_2ax + k_2by + c_2z$ and $L_{1,\alpha_1+1} = k_3ax + k_3by + c_3z$, then the common point $\mathbf{v}_\alpha$ is $(-kb, ka, 0)$ for some $k \neq 0$.
3. Only two of the three lines (P_{1,α_1+1}, P_{2,α_2+1} and P_{3,α_3+1}) are distinct. Let $\mathbf{v}_\alpha$ be the point of intersection of the these two lines. If the lines are parallel, then as in case 2, the point of intersection lies at infinity.

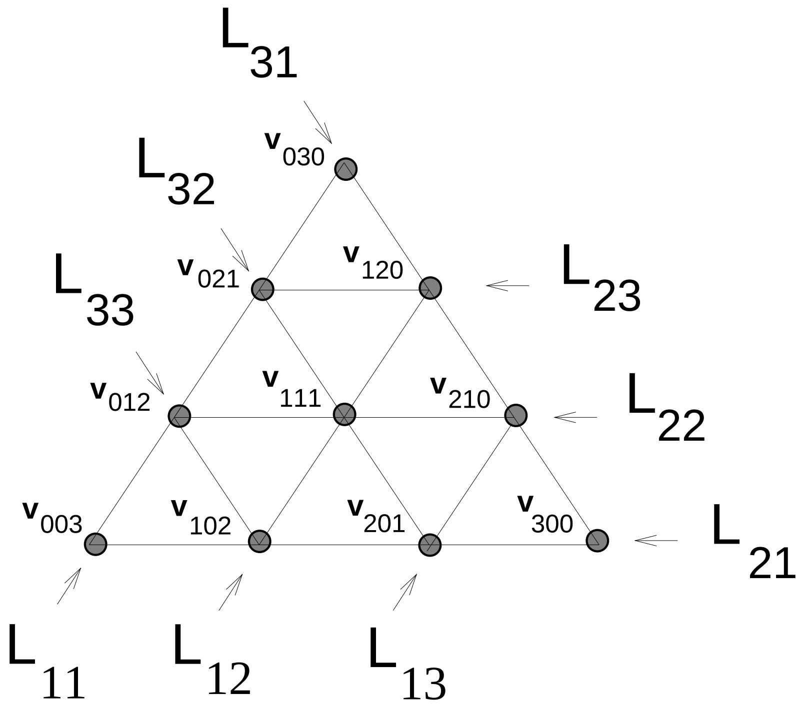

Figure 4.2: Geometric mesh of order 3 for Lagrange L-basis

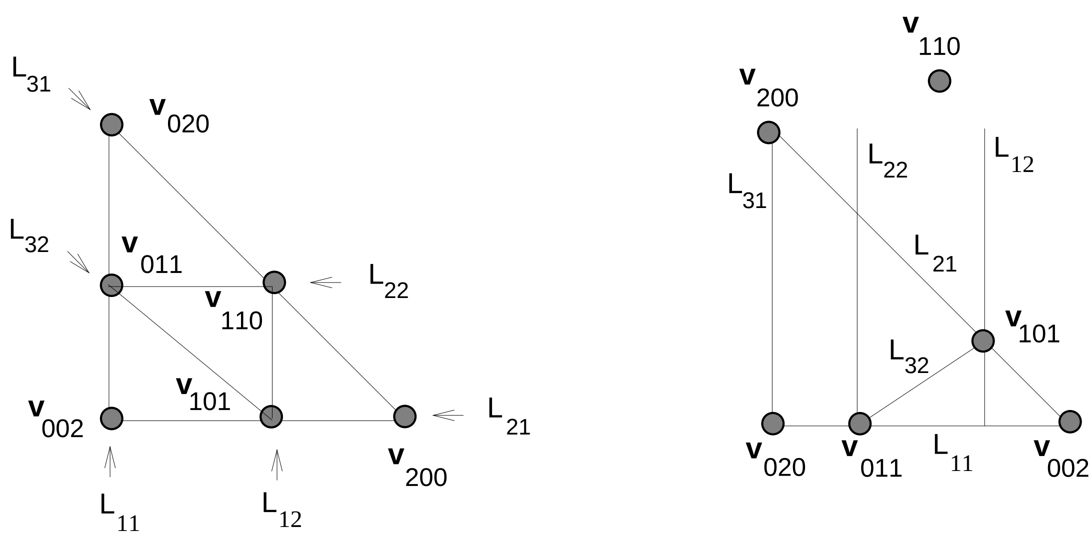

Figure 4.3: Geometric mesh of order 2 for Lagrange L-basis

4. One of the lines L_{k,α_k+1} lies at infinity. In this case the point of intersection of the lines $\{\ L_{1,\alpha_1+1},\ L_{2,\alpha_2+1},\ L_{1,\alpha_1+1}\ \}$ lies at infinity.

Observe that it is not possible to have all three lines the same because this would violate the linear independence condition on the knot-net of polynomials. More specifically, the linear dependence condition for $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha| = n$ and the linear independence condition for $(\alpha_1 - 1, \alpha_2, \alpha_3)$ imply that if the two lines L_{2,α_2+1} and L_{3,α_3+1} are same, then

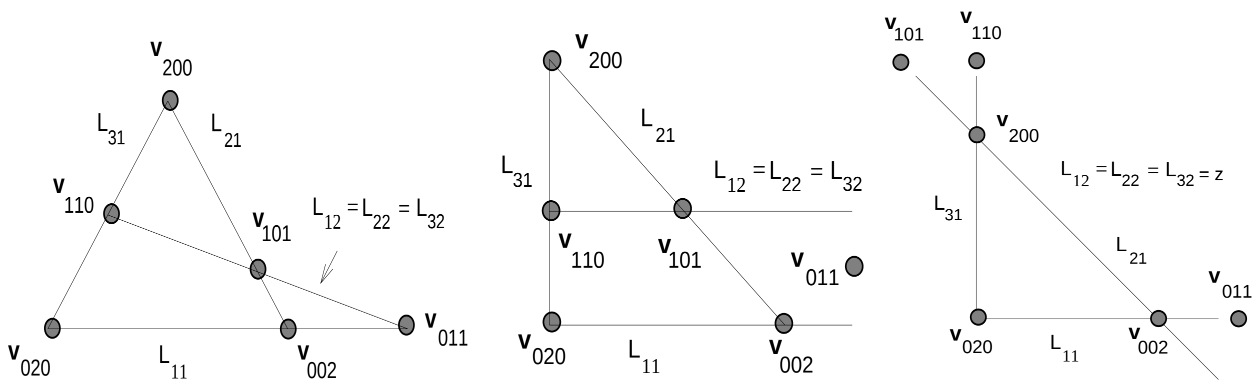

Figure 4.4: Natural lattice of order 2 for Lagrange L-basis

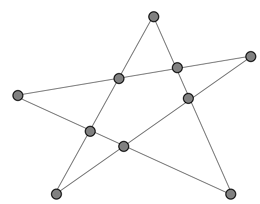

Figure 4.5: Natural lattice of order 3

α_1 must be zero. Therefore, if all three lines are the same, then it must be the case that $\alpha_i = 0$ for $i = 1, 2, 3$; that is $n = 0$, in which case there is only one L-basis function. This argument also shows that the condition that two lines are the same is very restrictive and can happen only if one of the three $\alpha_i = 0$. Such cases, however, do arise in practice as we shall see below.

If an affine polynomial $B(\mathbf{v}$ of degree n is represented with respect to an affine Lagrange $L - basis$, that is,

$$B(\mathbf{v}) = \sum_{|\alpha|=n} C_\alpha P_\alpha^n(\mathbf{v}),$$

the coefficients C_α represent, up to constant multiples, the value of the polynomial $B(\mathbf{v})$ at v_α, whenever v_α is not at infinity. More precisely, $B(\mathbf{v}_\alpha) = C_\alpha P_\alpha^n(\mathbf{v}_\alpha)$. When v_α is at infinity, as in the cases 2, 3 and 4 above, it can be verified easily, that the coefficients C_α represent, up to constant multiples, the directional derivative of $B(\mathbf{v})$ of order n in the direction of one of the parallel lines, that give rise to v_α as the common point of intersection. Observe that since $B(\mathbf{v})$ is a polynomial of degree n, its directional derivative of order n is a constant and therefore, it does not matter where it is evaluated.

Now we present certain point-line configurations that give rise to Lagrange L-bases. Figure 4.2 shows a configuration of lines in R^2 for which the dependency conditions are satisfied and all the lines are distinct and concurrent. The configuration of lines in Figure 4.2 also satisfy the linear independence condition for $(L_{1,\alpha_1+1}, L_{2,\alpha_2+1}, L_{3,\alpha_3+1})$, $0 \leq |\alpha| \leq n-1$, which is required to define a knot-net of polynomials. Figure 4.2 is an example of a *principal lattice* or *geometric mesh* [CY77] of order n, which can be described by three sets of n lines

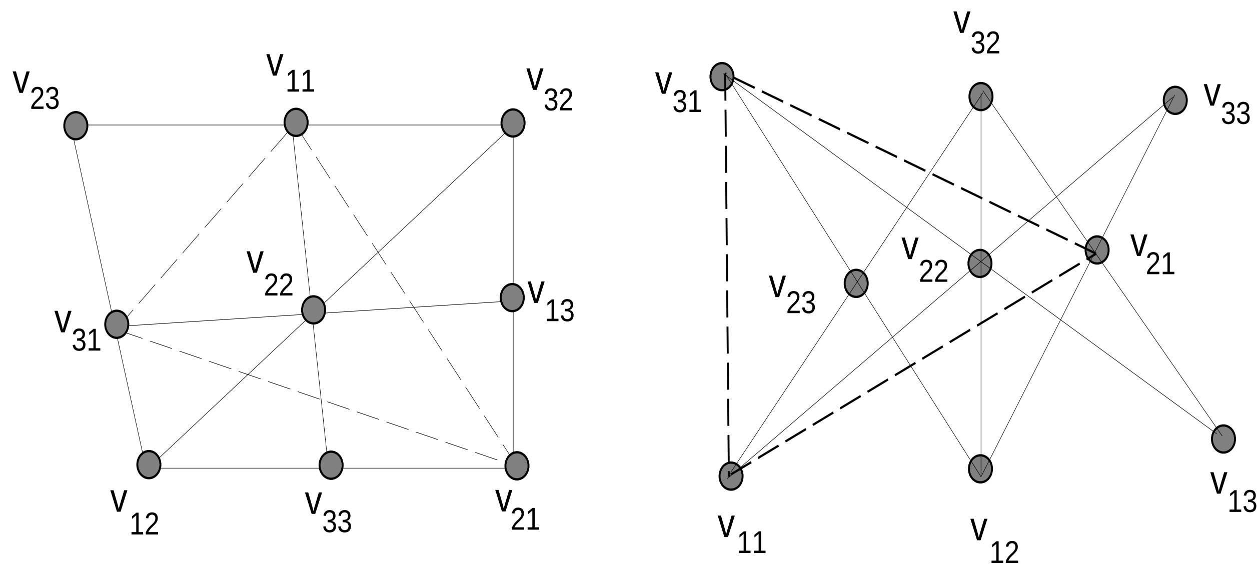

Figure 4.6: Dual geometric mesh of order 3 for power B-basis

$\{\{L_{1i}\}, \{L_{2j}\}, \{L_{3k}\}, 1 \le i, j, k \le n\}$ such that each set of three lines $\{L_{1,i+1}, L_{2,j+1}, L_{3,k+1}, i+j+k=n\}$ intersect at exactly one common point $\mathbf{v}_{ijk}$. It is clear from the above construction that every geometric mesh gives rise to a Lagrange L-basis.

Figure 4.3 shows some configurations of 6 lines and 6 points in the projective plane that give rise to a Lagrange L-basis. These are examples of geometric meshes of order 2. The right diagram of Figure 4.3 shows a configuration where one of the points is at infinity.

Figure 4.4 shows some configurations of 4 lines and 6 points in the projective plane that give rise to a Lagrange L-basis. In this case, two of the lines in every dependency condition are the same. These are examples of *natural lattices* [CY77] of order n, which are defined by $n+2$ lines in the projective plane such that the $\binom{n+2}{2}$ intersection points of these lines are all distinct. The left, middle and right diagrams of Figure 4.4 show configurations where 0, 1 and 3 points lie at infinity. Since every natural lattice of order n generates a Lagrange basis of degree n, it is natural to ask whether every natural lattice of order n gives rise to a Lagrange *L-basis* of degree n. Unfortunately, the answer is no. Figure 4.5 shows a natural lattice of order 3. It is easy to verify that it is not possible to realize the Lagrange basis corresponding to this configuration as an L-basis. Thus the Lagrange L-bases form a proper subset of the set of all bivariate Lagrange bases.

4.2.2 Power Bases

Let $\{\ \mathbf{u}_{1j},\ \mathbf{u}_{2j},\ \mathbf{u}_{3j},\ j = 1, \cdots, n\ \}$ be a knot-net of vectors. Suppose the vectors $(\mathbf{u}_{1,\alpha_1+1}, \mathbf{u}_{2,\alpha_2+1}, \mathbf{u}_{3,\alpha_3+1})$ are linearly dependent for $|\alpha| = n$, $0 \le \alpha_k \le n-1$. The corresponding B-basis is referred to as a power basis because, as we shall soon see, up to constant multiples every basis function is an n-th power of a linear polynomial.

Let $\mathbf{v}_{k,\alpha_k+1}$ represent the points in the projective plane corresponding to the vectors $\mathbf{u}_{k,\alpha_k+1}$. Then the linear dependency condition on the vectors $\mathbf{u}_{k,\alpha_k+1}$ means that the corresponding points $\mathbf{v}_{k,\alpha_k+1}$ are collinear in the projective plane. Let Q_α be the line defined by the three collinear points $\mathbf{v}_{k,\alpha_k+1}$ for $|\alpha| = n$, $0 \le \alpha_k \le n-1$ and let $q_\alpha = 0$ be the equation of the line Q_α. This construction gives rise to $\binom{n+2}{2} - 3$ lines corresponding to

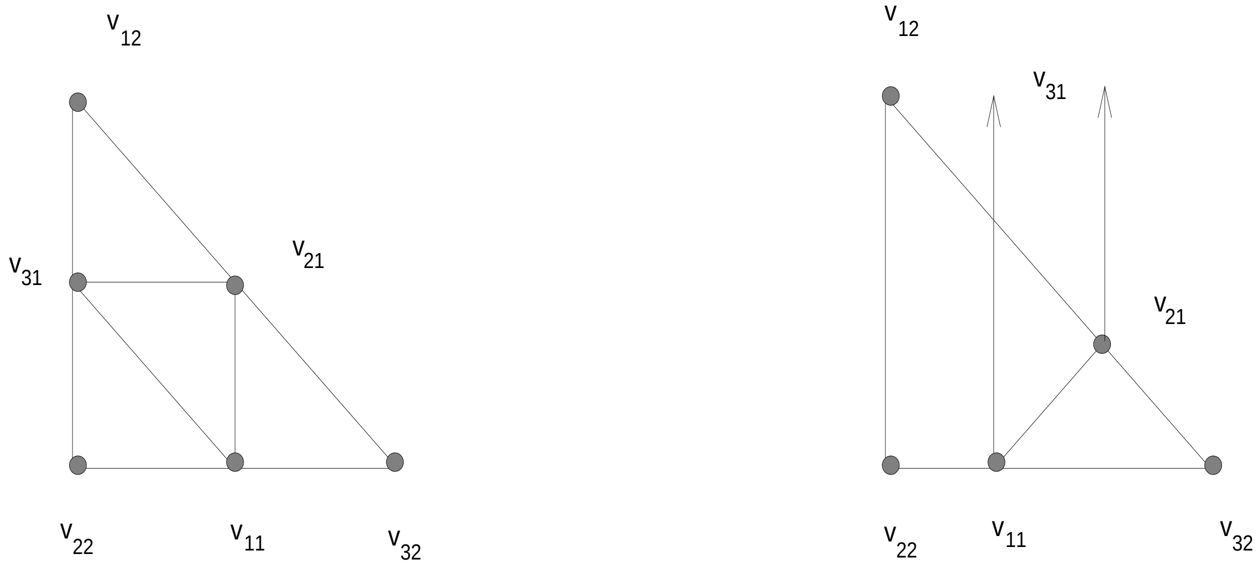

Figure 4.7: Dual geometric mesh of order 2 for power B-basis

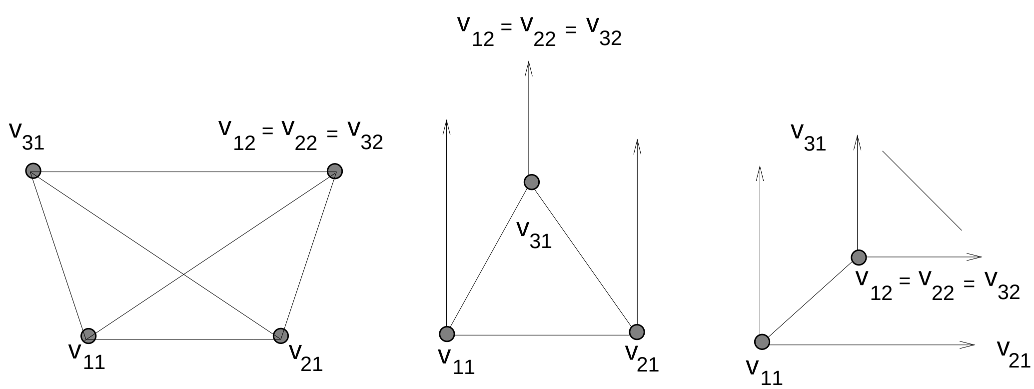

Figure 4.8: Dual natural lattice of order 2 for power B-basis

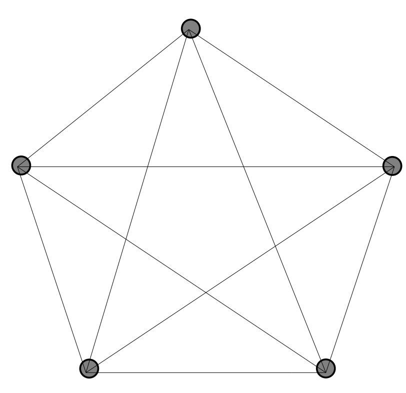

Figure 4.9: Dual natural lattice of order 3

the $\binom{n+2}{2} - 3$ dependency conditions. Now let us add 3 more lines. Define Q_{n00}, Q_{0n0}, and Q_{00n} to be the lines passing through the points $\mathbf{v}_{21}\mathbf{v}_{31}$, $\mathbf{v}_{11}\mathbf{v}_{31}$, and $\mathbf{v}_{11}\mathbf{v}_{21}$ respectively, and let q_{n00}, q_{0n0}, and q_{00n} be the equations of these lines. In the Appendix we give an inductive proof that the B-basis functions b^n_α for the knot-net $\mathbf{u}_{ij}$ are equal to $(q_\alpha)^n$ up to constant multiples. In the next section, we shall give a much simpler proof of this fact based on the duality between the Lagrange and power bases.

Figure 4.6 shows two configuration of points in R^2 for which the dependency conditions are satisfied because the points $\mathbf{v}_{k,\alpha_k+1}$ for $|\alpha| = n$ are collinear. The configuration of points in Figure 4.6 also satisfy the linear independence condition for $\mathbf{u}_{k,\alpha_k+1}, k = 1, 2, 3, |\alpha| \leq n-1$, which is required to define a knot-net of vectors. This figure is an example of a *dual principal lattice* or *dual geometric mesh* of order n, which is defined by $3n$ distinct points $\{\mathbf{v}_{1j}, \mathbf{v}_{2j}, \mathbf{v}_{3j}, j = 1, \cdots, n\}$ such that each set of three points $\{\mathbf{v}_{1,i+1}, \mathbf{v}_{2,j+1}, \mathbf{v}_{3,k+1}, i+j+k = n\}$ is collinear and defines the line Q_{ijk}. The seven lines defined by the dependency conditions are shown as dark lines while the remaining three add-on lines are shown as dotted lines. It is clear from the above construction that every dual geometric mesh gives rise to a power B-basis.

Figure 4.7 shows examples of point-line configurations with 6 points and 6 lines that give rise to power B-bases. These are examples of dual geometric meshes of order 2. The right diagram of Figure 4.7 shows a configuration where one of the points lie at infinity and this is represented by a vector in the affine plane.

Figure 4.8 shows some configurations of 6 lines and 4 points in the projective plane that give rise to a power B-basis. These are examples of *dual natural lattices* of order 2. A *dual natural lattice* of order n is defined by $n+2$ distinct points and $\binom{n+2}{2}$ distinct lines joining these points. The left, middle and right diagrams of Figure 4.8 show configurations of points and lines, where 0, 1 and 2 points lie at infinity and these are represented by vectors in the affine plane. Since every dual natural lattice of order n generates a power basis of degree n, it is natural to ask whether every dual natural lattice of order n gives rise to a power *B-basis* of degree n. Unfortunately, the answer is no. It is easy to verify by exhaustive enumeration that the configuration of points and lines corresponding to the dual natural lattice of order 3 shown in Figure 4.9 cannot be realized as a B-basis. A simpler proof based on duality will be given at the end of next section 4.2.3. Thus the power B-bases form a proper subset of the set of all bivariate power bases.

4.2.3 Duality

Let a Lagrange L-basis be defined by a knot-net $\mathcal{L}$ of polynomials $\{L_{ij}, L_{2j}, L_{3j}, j = 1, \cdots, n\}$ as in Section 4.2.1, and let the $\binom{n+2}{2}$ points corresponding to this Lagrange L-basis be denoted by $\mathbf{v}_\alpha$. Let the dual B-basis be defined by the knot-net $\mathcal{U}$ of vectors $\{\mathbf{u}_{ij}, \mathbf{u}_{2j}, \mathbf{u}_{3j}, j = 1, \cdots, n\}$ under the knot net correspondence $(a, b, c) \leftrightarrow ax + by + cz$ defined in Section 3.1 so that $L_{ij}(\mathbf{u}) = \mathbf{u} \cdot \mathbf{u}_{ij}$. It is clear that both the linear independence conditions and the linear dependence conditions are preserved under this correspondence. In particular, the linear dependency condition or the collinearity condition on a set of three points used for defining B-bases corresponds to the linear dependency condition or the concurrency condition on the corresponding set of three lines used for defining L-bases. Therefore, the dual B-basis is a power basis as defined in Section 4.2.2.

The Cavaretta-Micchelli identity mentioned in Section 3.1 provides a very simple proof that the B-basis dual to a Lagrange L-basis is a power basis, that is, that every element of the B-basis is a n-th power of a linear polynomial. Indeed given a Lagrange L-basis, it was verified in Section 4.2.1 that the L-basis functions $\{l_\alpha^n\}$ satisfy the relation $l_\alpha^n(\mathbf{v}_\beta) = l_\alpha^n(\mathbf{v}_\alpha)\delta_{\alpha\beta}$. Substituting this identity into the Cavaretta-Micchelli identity, we obtain $b_\alpha^n(\mathbf{u}) = \frac{(\mathbf{v}_\alpha \cdot \mathbf{u})^n}{l_\alpha^n(\mathbf{v}_\alpha)}$, which establishes that up to constant multiples each element of the dual B-basis is an n-th power of a linear polynomial. Using the definition of the L-basis functions given in Equation 2.1 together with the fact that by duality $q_\alpha = \mathbf{v}_\alpha \cdot \mathbf{u}$, we can also rewrite the power B-basis functions as

$$b_\alpha^n(\mathbf{u}) = \frac{(q_\alpha)^n}{\prod_{j=1,\cdots,\alpha_i;i=1,2,3} q_\alpha(\mathbf{u}_{ij})}.$$

Notice that the Cavaretta-Micchelli identity and the generalized de Boor-Fix formula hold for all bivariate Lagrange and power bases, even though these bases may not be L-bases and B-bases respectively. The argument in the preceding paragraph can be used to establish this general duality between bivariate Lagrange and power bases.

To appreciate the geometry of this correspondence, notice that a Lagrange L-basis of degree n is defined, in general, by $3n$ lines while a power B-basis of degree n is defined, in general, by $3n$ points. However, these lines (in case of the Lagrange L-basis) and points (in case of the power B-basis) need not be distinct. Such is the case, for example, with the natural lattice and dual natural lattice configurations, where certain lines in case of the Lagrange L-basis and certain points in the case of power B-basis do coincide.

The geometric mesh configuration of order 3 for a Lagrange L-basis depicted in Figure 4.2 consisting of 9 distinct lines and 10 distinct points is dual to the dual geometric mesh configuration of order 3 for a power B-basis depicted in Figure 4.6 consisting of 9 distinct points and 10 distinct lines.

Similarly, the geometric mesh configuration of order 2 for a Lagrange L-basis depicted in Figure 4.4 consisting of 6 distinct lines and 6 distinct points is dual to the dual geometric mesh configuration of order 2 for a power B-basis depicted in Figure 4.8 consisting of 6 distinct points and 6 distinct lines. The distinction between different cases as to whether certain points lie at infinity or whether certain lines are parallel disappears in projective space. Figure 4.4 and Figure 4.8 are self-dual. is self-dual. In a self-dual configuration, the number of lines must be equal to the number of points. Since $3n = \binom{n+2}{2}$ only for $n = 1, 2$, these are the only situations where the geometric mesh configuration is self-dual. Figure 4.6 shows the dual geometric mesh for $n = 3$ which is not self-dual.

The natural lattice configuration of order 2 depicted in Figure 4.3 consisting of 4 lines and 6 points for the Lagrange L-basis is dual to the dual natural lattice configuration of 6 lines and 4 points for the power B-basis depicted in Figure 4.7. Finally, the natural lattice configuration depicted in Figure 4.5 consisting of 5 lines and 10 points cannot be realized as a Lagrange L-basis. Therefore by duality the dual natural lattice configuration of 10 lines and 5 points depicted in Figure 4.9 cannot be realized as a power B-basis.

4.3 Duality between Newton and Newton Dual Bases

This section establishes that certain subclasses of bivariate Newton bases can be realized as special cases of L-bases. We then introduce the Newton dual bases and investigate the duality between the Newton and Newton dual bases.

4.3.1 Newton Bases

Suppose the following restriction is imposed on the knot-net $\mathcal{L}$ of homogeneous polynomials: $L_{1i} = a_{1i}x + b_{1i}y + c_{1i}z$, $L_{2i} = a_{2i}x + b_{2i}y + c_{2i}z$, and $L_{3i} = a_3x + b_3y + c_3z$. In other words, one of the three sets of knots consists of one and the same polynomial. The corresponding L-basis will be referred to as the generalized bivariate homogeneous Newton L-basis. We shall be interested here in the special case where $L_{3i} = z$; the corresponding L-basis will be referred to as the bivariate homogeneous Newton L-basis. Observe that the multinomial basis is a special case of Newton basis. By choosing the polynomials $L_{1i} = x - a_iz$, $L_{2i} = y - b_jz$, $L_{3i} = z$, and dehomogenizing, we obtain the following corresponding affine Newton basis:

$$l_\alpha^n = \prod_{i=1}^{\alpha_1}(x - a_i)\prod_{j=1}^{\alpha_2}(y - b_j).$$

For example when $n = 2$ this construction yields the basis functions: 1, $(x - a_1)$, $(x - a_1)(x - a_2)$, $(y - b_1)$, $(y - b_1)(y - b_2)$ and $(x - a_1)(y - b_1)$. Therefore, the affine Newton basis for surfaces defined here is a generalization of the affine Newton basis for curves.

To justify the terminology Newton basis, we are going to establish that these Newton L-bases are special cases of the bivariate Newton bases defined by Gasca [Gas90]. Gasca starts with a particular set of lines and points and associates a Newton basis to this point-line configuration. In contrast, our construction proceeds in the opposite direction.

We have other stronger incentives for establishing this connection. We plan to construct certain point-line configurations and associated point and derivative interpolation problems, which give rise to Newton L-bases in a natural way. To be more precise: to each Newton L-basis, we wish to associate an interpolation problem for point and derivative data with the following properties: (i) there exists a unique solution to the general interpolation problem and (ii) the coefficients a_α of the interpolant $L(\mathbf{u}) = \sum_{|\alpha|=n} a_\alpha l_\alpha^n$ expressed in the Newton L-basis are the solutions of a lower triangular system of linear equations. This task is complicated, however, by the fact that given a suitable point-line configuration, the associated point and derivative interpolation problem is not unique. This non-uniqueness is intrinsic to the bivariate Newton basis and is true as well for the univariate Newton basis.

To gain some insight into this important point, we explain the nature and cause of non-uniqueness in the case of curves by presenting a simple example. To the univariate affine Newton basis of degree 2 given by 1, $(x - a_1)$ and $(x - a_1)(x - a_2)$, $a_1 \neq a_2$, one can associate a point interpolation problem at a_1 and a_2, but for the third interpolation condition one can choose any arbitrary point a_3 or in fact even the derivative at a_2. Indeed if $f(x) = c_0 + c_1(x - a_1) + c_2(x - a_1)(x - a_2)$, then $c_0 = f(a_1)$, and $c_1 = f[a_2, a_1]$, where

$$f[a_2, a_1] = \frac{f(a_2) - f(a_1)}{(a_2 - a_1)}$$

is the usual divided difference. More interestingly, $c_2 = f[a_3, a_2, a_1]$ where

$$f[a_3, a_2, a_1] = \frac{f[a_3, a_2] - f[a_2, a_1]}{(a_3 - a_1)} \quad if \quad a_3 \neq a_2,$$

$$f[a_2, a_2, a_1] = \frac{f'(a_2) - f[a_2, a_1]}{(a_2 - a_1)} \quad if \quad a_3 = a_2,$$

where $f'(a_2)$ denotes the first derivative of f at a_2. Similarly, if the Newton basis is 1, $(x - a_1)$ and $(x - a_1)^2$, then $c_0 = f(a_1)$, $c_2 = f'(a_1)$, and $c_2 = f[a_2, a_1, a_1]$, where a_2 is any arbitrary point including a_1.

This freedom in choosing the interpolation problem carries over to the bivariate setting. Although Gasca [Gas90] observes that there is some freedom, his construction does not clarify the role of freedom in choosing the lines and points. For our purposes, it is essential to explore the nature of this non-uniqueness in order to specify certain interesting point-line configurations associated with Newton L-bases.

To associate an interpolation problem with a Newton L-basis, we first introduce a set of points. To this purpose, observe that the linear independence conditions on the knot-net of polynomials imply that the lines L_{1,α_1+1} and L_{2,α_2+1} are distinct and non-parallel for $0 \leq \alpha_1+\alpha_2 \leq n-1$. Let $\mathbf{v}_{\alpha_1,\alpha_2,\alpha_3} = L_{1,\alpha_1+1} \cap L_{2,\alpha_2+1}$ for $0 \leq \alpha_1+\alpha_2 \leq n-1$. These points could be distinct or the same depending upon the lines themselves, but, in any event we get $\frac{1}{2}n(n+1)$ points counted with appropriate multiplicity. Next we introduce an additional $n+1$ points for a total of $\binom{n+2}{2}$ points again counted with appropriate multiplicity. The choice of the remaining $n+1$ points $\mathbf{v}_{\alpha_1,\alpha_2,0}$ is more subtle and incorporates the freedom of choice discussed in the previous paragraph. Choose $\mathbf{v}_{n00}$ and $\mathbf{v}_{0n0}$ to be arbitrary points on the line L_{21} and L_{11} respectively. For the remaining points with $\alpha_1 + \alpha_2 = n$ and $0 \leq \alpha_1, \alpha_2 \leq n-1$, there are three types of choices: (i) symmetric case: let $\mathbf{v}_{\alpha_1,\alpha_2,0} = L_{1,\alpha_1+1} \cap L_{2,\alpha_2+1}$ whenever L_{1,α_1+1} and L_{2,α_2+1} are distinct; otherwise, if they are the same line, choose any point on this line, (ii) non-symmetric case 1: let $\mathbf{v}_{\alpha_1,\alpha_2,0}$ be any point on the line L_{1,α_1+1} and (iii) non-symmetric case 2: let $\mathbf{v}_{\alpha_1,\alpha_2,0}$ be any point on the line L_{2,α_1+1}. This freedom in choosing the $(n+1)$ points can also be described by selecting additional lines $F_1, \cdots, F_{n+1}$, $F_i \neq L_{1,i}$ so that $\mathbf{v}_{\alpha_1,\alpha_2,0} = L_{1,\alpha_1+1} \cap F_{\alpha_1+1}$ or alternatively by selecting additional lines $F_1, \cdots, F_{n+1}$, $F_j \neq L_{2,j}$ so that $\mathbf{v}_{\alpha_1,\alpha_2,0} = L_{2,\alpha_2+1} \cap F_{\alpha_2+1}$. Thus we can associate a total of $\binom{n+2}{2}$ points counted with appropriate multiplicity to a Newton L-basis defined by $2n$ lines. This point-line configuration associated with a Newton L-basis is a subclass of the point-line configurations that form the starting point for the construction of Newton bases defined by Gasca [Gas90]. With this associated point-line configuration, it can be readily verified that the Newton L-bases defined here can be realized as special cases of the bivariate Newton bases defined by Gasca.

Now we are in a position to describe the interpolation system associated with this Newton L-basis. Our procedure is exactly the same as in [Gas90]. Let s_α be the number of functions in the set $\{L_{1,1}, \cdots, L_{1,\alpha_1+1}, L_{2,1}, \cdots, L_{2,\alpha_2+1}\}$ that vanish at v_α and coincide with L_{1,α_1+1} up to constant factors. Let t_α be the number of functions in the set $\{L_{1,1}, \cdots, L_{1,\alpha_1+1}, L_{2,1}, \cdots, L_{2,\alpha_2+1}\}$ that vanish at v_α and *do not* coincide with L_{1,α_1+1} up to constant factors. The associated interpolation problem is to interpolate the following point and derivative information:

$$\frac{\partial^{s_\alpha+t_\alpha} f(\mathbf{v}_\alpha)}{\partial^{s_\alpha} L_{1,\alpha_1+1} \partial^{t_\alpha} L_{2,\alpha_2+1}},$$

where $\frac{\partial f}{\partial L} = b\frac{\partial f}{\partial x} - a\frac{\partial f}{\partial y}$, when $L = ax + by + c$.

It is not too difficult to prove that this interpolation problem has a unique solution and that the interpolant expressed in terms of the Newton L-basis can be found by solving a lower triangular system of linear equations. The proof of this fact is also described by Gasca [Gas90] and is therefore omitted here. In fact, the coefficients of the solution can be interpreted as the generalization of divided differences to higher dimensions. Further discussion

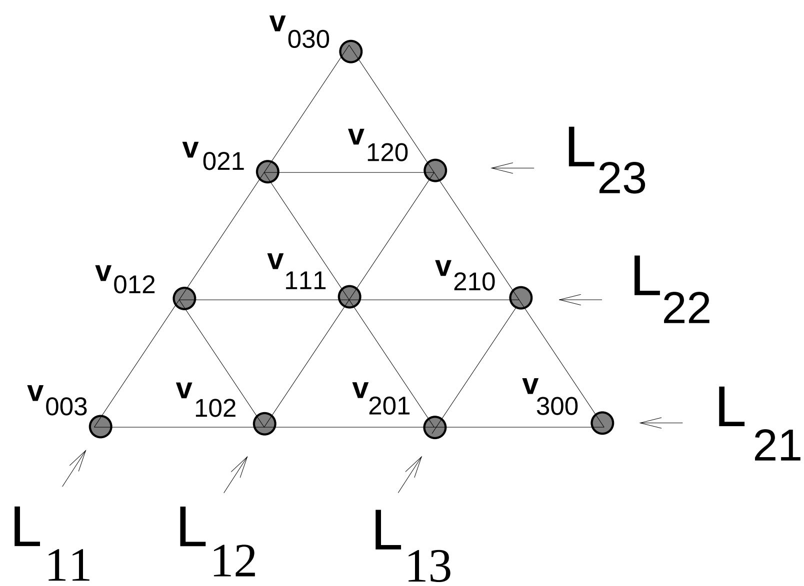

Figure 4.10: Point interpolation using Newton L-basis

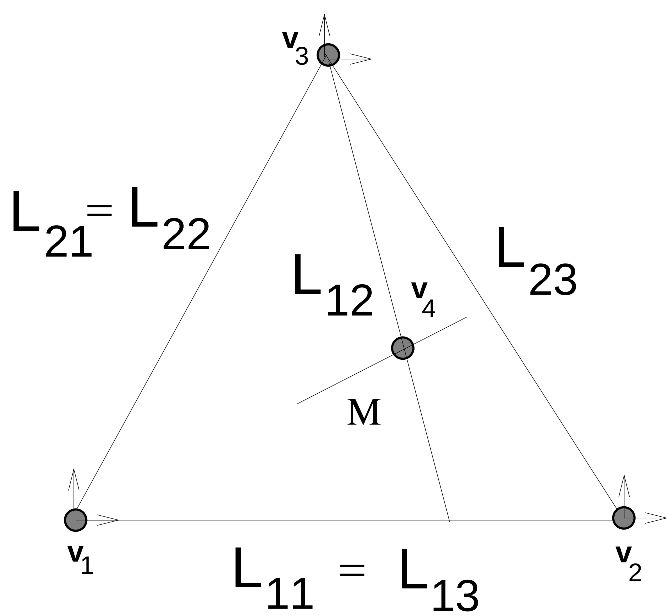

Figure 4.11: Hermite interpolation using Newton L-basis

of the extremely important role the Newton bases play in multivariate interpolation and approximation can be found in [Gas90].

The configurations of lines and points corresponding to Newton L-bases are very flexible. By choosing two sets of parallel lines $L_{1,i}$ and $L_{2,j}$ as shown in Figure 4.10, selecting the symmetric choice of the associated interpolation problem, and picking the points $\mathbf{v}_{300}$ and $\mathbf{v}_{030}$ as indicated in Figure 4.10, it is clear that every geometric mesh gives rise to a Newton L-basis. Recall from Section 4.2 (Figure 4.2) that the same configuration also gives rise to a Lagrange L-basis.

More interestingly, every natural lattice of order n also gives rise to a Newton L-basis. Figure 4.5 shows a natural lattice of order 3. This lattice gives rise to the Newton L-basis by choosing $L_{1,i} = L_i$ for $1 \le i \le n$, $L_{2,j} = L_{n+3-j}$, $1 \le j \le n$, selecting the symmetric choice, and picking the point $\mathbf{v}_{0n0}$ on L_1 as $L_1 \cap L_2$ and the point $\mathbf{v}_{n00}$ on L_{n+2} as $L_{n+2} \cap L_{n+1}$.

It is not yet clear whether or not every configuration of lines and points that gives rise to a Lagrange L-basis also gives rise to a Newton L-basis. More interestingly, one can ask whether a configuration of lines and points satisfying the GC condition or HGC condition [Gas90] always gives rise to a Newton L-basis. The HGC conditions, in particular, generalize the Hermite interpolation conditions to higher dimensions.

A popular Hermite problem corresponding to point interpolation at four points $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, and the two first order partial derivatives at the three points $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ is depicted in Figure 4.11. Even for this simple Hermite case, it is non-trivial to demonstrate that it can be realized as a Newton L-basis. The choice $L_{11} = L_{13}$, $L_{12} = L_{23}$ as shown in Figure 4.11 yields the Hermite interpolation problem by picking the following lines $F_1 = L_{21}$, $F_2 = L_{21}$, $F_3 = M$ and $F_4 = L_{12}$, considering the intersection points $F_i \cap L_{1i}$ for $i = 1, 2, 3$ and the intersection point F_4 with L_{21}. This is a non-symmetric choice. By enumerating all the possibilities, one can verify that it is not possible to generate a Newton L-basis corresponding to this interpolation problem with any symmetric choice.

4.3.2 Newton Dual Bases

The homogeneous generalized Newton Dual basis is defined as the B-basis functions obtained by imposing the following restrictions on the knot-net $\mathcal{U}$ of vectors: $\mathbf{u}_{1i} = (a_{1i}, b_{1i}, c_{1i})$, $\mathbf{u}_{2i} = (a_{2i}, b_{2i}, c_{2i})$, and $\mathbf{u}_{3i} = (a_3, b_3, c_3)$. In other words, the generalized Newton dual basis is obtained by restricting one of the three sets of vectors in the knot-net to contain exactly one element. The homogeneous Newton dual basis for surfaces defined here is a generalization of the Newton dual basis for curves [BG93]. Observe that the Bézier, multinomial, and BM B-bases are special cases of the Newton dual basis. Another important subclass of the generalized Newton dual basis is obtained by imposing the following restrictions on the knot-net: $\mathbf{u}_{1i} = (a_{1i}, b_{2i}, 1)$, $\mathbf{u}_{2i} = (1, 0, 0)$, and $\mathbf{u}_{3i} = (0, 1, 0)$. One interesting and useful property of this Newton dual basis is that in the up recurrence relation for the B-basis defined in Section 2.2 by Equation 2.2, the labels $h_{k,\alpha}(\mathbf{u})$ do not involve any divisions. Indeed, if $\mathbf{u} = (x, y, z)$, then $h_{1,\alpha} = z$, $h_{2,\alpha}(\mathbf{u}) = x - a_{1,\alpha_1+1}z$, $h_{3,\alpha}(\mathbf{u}) = y - b_{1,\alpha_1+1}z$. This property of the Newton dual basis can be applied to minimize divisions and simplify computations in change of basis algorithms.

For the sake of completeness, we describe an explicit expression for these special Newton dual basis functions. Let $L_i = y - b_{1i}z$ and $M_i = x - a_{1i}z$. Then the dual basis functions b_α^n are given by:

$$b_\alpha^n = \sum L_1^{\alpha_{21}} \cdots L_{\alpha_1+1}^{\alpha_{2,\alpha_1+1}} M_1^{\alpha_{31}} \cdots M_{\alpha_1+1}^{\alpha_{3,\alpha_1+1}} z^{\alpha_1},$$

where the sum is taken over all $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha| = n$ and $\alpha_2 = \alpha_{21} + \cdots + \alpha_{2,\alpha_1+1}$, $\alpha_3 = \alpha_{31} + \cdots + \alpha_{3,\alpha_1+1}$, and $\alpha_{ij} \ge 0$. The derivation is straightforward from the definition, although the bookkeeping is somewhat tedious.

4.3.3 Duality

Under the knot-net correspondence it is clear from the construction that the generalized Newton basis is dual to the generalized Newton dual basis. However it is *not* true that the Newton basis is dual to the Newton dual basis. Nevertheless, it is these special cases of the Newton basis and Newton dual basis that turn out to be the most useful in practical situations and hence the terminology. The duality here is similar to the duality we encountered for uniform bases, where a uniform L-basis is dual to a uniform B-basis, although a Bézier basis could be dual to either a Bézier, a multinomial, or a hybrid BM basis. Similarly, the generalized Newton basis is dual to a generalized Newton dual basis, although a Newton basis itself may not necessarily be dual to a Newton dual basis.

5. Applications of Duality

There are many applications of duality between B-bases and L-bases. We can use geometric duality to show that a particular point-line configuration can (cannot) represent the knot-net for a B-basis (L-basis) by showing that the dual configuration can (cannot) represent the knot-net for the dual L-basis (B-basis). We used this argument in Section 4.2.3 to conclude that the dual natural lattice of order 3 (Figure 4.9) cannot represent the knot-net of a power B-basis because we already knew that the natural lattice of order 3 (Figure 4.5) does not represent the knot-net of any Lagrange L-basis.

We can also use algebraic or de Boor-Fix duality to great effect. By applying algebraic duality, we can show that many formulas and algorithms for B-bases map to dual formulas and algorithms for L-bases and vice-versa. Thus once we develop a formula or algorithm for one type of basis we can often obtain, almost for free, a dual formula or algorithm for the dual basis. Formulas and algorithms for change of bases [LG95], evaluation [LG94c], differentiation [LG94b], degree elevation [LG94b], and subdivision [LG94a] each have dual analogues for B-bases and L-bases. This observation allows us to develop a formula or algorithm for whichever scheme is easier to analyze and then map this to a dual formula or algorithm for the dual scheme.

A general change of basis algorithm for B-bases is easy to derive via blossoming [LG95]. By de Boor-Fix duality, we can use this procedure to construct dual change of basis algorithm for L-bases [LG95]. As an application of the constructions in this paper, in the next section we apply this algorithm to convert a bivariate polynomial from a Lagrange representation to a Bézier representation. We also observe that the reverse transformation from Bézier to Lagrange form yields a fast evaluation algorithm for Bézier patches and hence as well for arbitrary B-patches and L-patches.

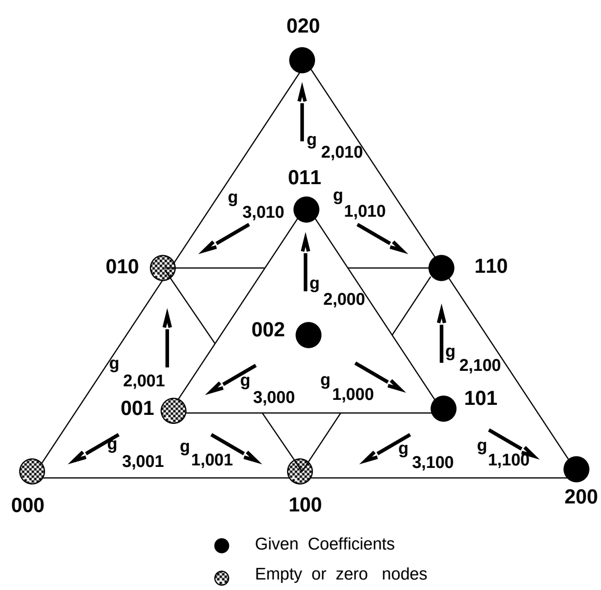

Figure 5.1: Labeling of the tetrahedron

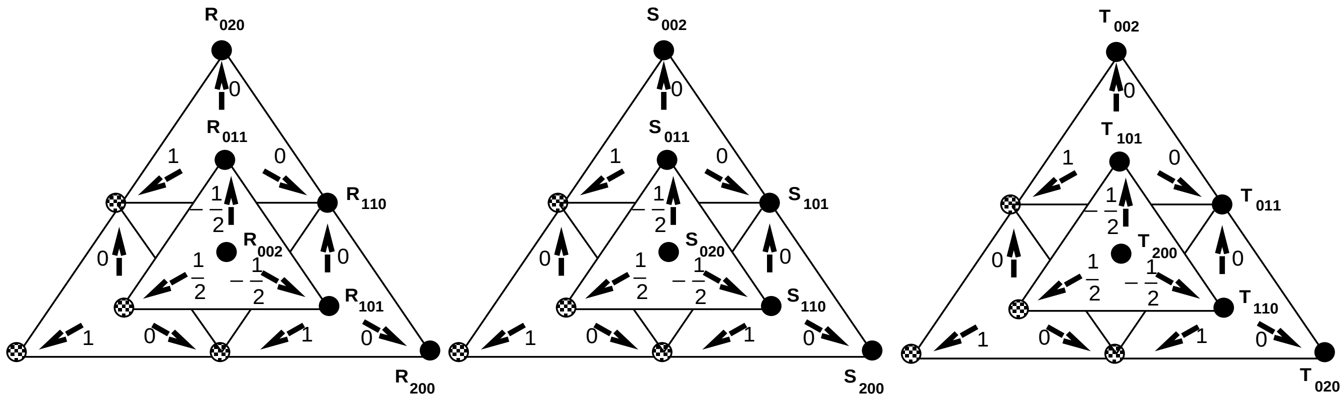

Figure 5.2: Change of basis from Lagrange L-basis to Bézier L-basis

5.1 A Change of Basis Algorithm for L-bases

The computational complexity of general change of basis algorithms from one bivariate polynomial basis of degree n to another bivariate polynomial basis of degree n using matrix multiplication is, in general, $O(n^4)$. Using blossoming and duality, we have derived change of basis algorithms with computational complexity $O(n^3)$ between any two B-bases, any two L-bases, and between any B-basis and any L-basis [LG95]. In this work we have demonstrated that certain bivariate Lagrange bases and Newton bases can be realized as L-bases and that certain power bases and Newton dual bases can be realized as B-bases. As a consequence, these change of basis algorithms can now be applied to convert between Bézier, multinomial, Lagrange, power, Newton, and Newton dual bases.

We shall now describe a specific example of a change of basis algorithm from a Lagrange L-basis to a Bézier L-basis to illustrate the general procedure.

Suppose we are given the coefficients R_α of a quadratic polynomial L with respect to the L-basis $\{l_\alpha^n\}$ defined by the knot-net $\mathcal{L} = \{\{L_{1j}\}, \{L_{2j}\}, \{L_{3j}\}, j = 1, 2\}$, where

$$\begin{array}{ll} L_{11} = x & ; L_{12} = x - \frac{1}{2} \quad ; \\ L_{21} = y & ; L_{22} = y - \frac{1}{2} \quad ; \\ L_{31} = 1 - x - y & ; L_{32} = \frac{1}{2} - x - y \ . \end{array}$$

The Lagrange L-basis is then given by $l_{200}^2 = x(x - \frac{1}{2})$, $l_{020}^2 = y(y - \frac{1}{2})$, $l_{002}^2 = (1 - x - y)(\frac{1}{2} - x - y)$, $l_{110}^2 = xy$, $l_{101}^2 = x(1 - x - y)$, and $l_{011}^2 = y(1 - x - y)$. The point-line configuration associated with this Lagrange L-basis is shown in the left diagram of Figure 4.3.

We would like to compute the coefficients U_α of this polynomial L with respect to the Bézier L-basis $\{p_\alpha^n\}$ defined by another knot-net $\mathcal{M} = \{\{M_{1j}\}, \{M_{2j}\}, \{M_{3j}\}, j = 1, 2\}$, where

$$\begin{array}{ll} M_{11} = x & ; M_{12} = x \quad ; \\ M_{21} = y & ; M_{22} = y \quad ; \\ M_{31} = 1 - x - y & ; M_{32} = 1 - x - y \ . \end{array}$$

The Bézier L-basis is then given by $p_{200}^2 = x^2$, $p_{020}^2 = y^2$, $p_{002}^2 = (1 - x - y)^2$, $p_{110}^2 = xy$, $p_{101}^2 = x(1 - x - y)$, and $p_{011}^2 = y(1 - x - y)$.

To describe the change of basis algorithm, we will construct three tetrahedra. We first explain the labeling scheme for these tetrahedra. For each tetrahedron, $\frac{(3-i)(4-i)}{2}$ nodes are placed at the i-th level of the tetrahedron for $i = 0, 1, 2$ and the nodes along one of the lateral faces are indexed by α for $|\alpha| = 2$. An arrow is placed pointing downward from a node α at i-th level to the three nodes $\alpha + e_1 - e_3$, $\alpha + e_2 - e_3$ and $\alpha - e_3$ at $(i-1)$-st level directly below it. This labeling scheme for the nodes is shown in Figure 5.1. Values, referred to as *labels*, are placed along the arrows. The labels are indexed as $g_{k,\alpha}$ for $k = 1, 2, 3$ and $|\alpha| = 0, 1, 2$ for an arrow from a node $(\alpha_1, \alpha_2, 2 - |\alpha|)$ at the $|\alpha|$-th level to the three nodes below it. This labeling scheme for the labels and the arrows is also shown in Figure 5.1.

For the first tetrahedron the known coefficients R_α with $|\alpha| = 2$ are placed at the nodes along one of the lateral faces of the tetrahedron as depicted in the first diagram of Figure 5.2. The labels $g_{k,\alpha}$ are computed as follows: for $|\alpha| = 0, 1$, let $i = 2 - |\alpha|$; then

$$L_{3i} = g_{1,\alpha} L_{1,\alpha_1+1} + g_{2,\alpha} L_{2,\alpha_2+1} + g_{3,\alpha} M_{3,\alpha_3+1}.$$

Thus finding $g_{k,\alpha}$ amounts to solving a 3×3 system of linear equations. For our example, the labels are: $g_{1,100} = g_{2,100} = -\frac{1}{2}$, $g_{3,100} = \frac{1}{2}$, $g_{3,100} = g_{3,010} = g_{3,001} = 1$. The rest of the labels are zero. These labels are shown in the first diagram of Figure 5.2. The computation is now carried out as follows. At the start all the nodes at all levels of the pyramid are empty or zero other than the nodes α with $|\alpha| = 2$, where the coefficients R_α are placed. The empty or zero nodes are shown as hatched circles in Figures 5.1 and 5.2. The computation starts at the apex of the tetrahedron and proceeds downwards. A value at any empty node is computed by multiplying the label along each arrow that enters the node by the value of the node from which the arrow emerges and adding the results. A value at any non-empty node is computed by applying the same procedure and simply adding the value already at that node. After the computation is complete, the new coefficients $S_{\alpha+(2-|\alpha|)e_3}$ emerge on the nodes α at the base triangle. These coefficients are as follows: $S_{200} = R_{200}$, $S_{110} = R_{110}$, $S_{020} = R_{020}$, $S_{101} = -\frac{1}{2} R_{002} + R_{101}$, $S_{011} = -\frac{1}{2} R_{002} + R_{011}$, $S_{002} = \frac{1}{2} R_{002}$. These new coefficients now express the polynomial L with respect to the L-basis defined by the knot-net $\{\{L_{1j}\}, \{L_{2j}\}, \{M_{3j}\}, j = 1, 2\}$.

We now repeat the above procedure with a second tetrahedron, where the coefficients S_α are placed at the nodes α with $|\alpha| = 2$ as shown in the middle diagram of Figure 5.2. The labels on the tetrahedron are permuted from (i, j, k) to (i, k, j) because we now wish to retain the polynomial M_{3j} and replace the polynomials L_{2j} by M_{2j}. The labels $g_{k,\alpha}$ are now computed as follows: For $|\alpha| = 0, 1$, let $i = 2 - |\alpha|$; then

$$L_{2i} = g_{1,\alpha} L_{1,\alpha_1+1} + g_{2,\alpha} M_{2,\alpha_2+1} + g_{3,\alpha} M_{3,\alpha_3+1}.$$

These labels are also shown in the middle diagram of Figure 5.2 and in our special case turn out to be the same as in the first tetrahedron. After the computation is complete, the new coefficients T_α emerge on the nodes at the base triangle. These coefficients are as follows: $T_{200} = S_{200}$, $T_{110} = -\frac{1}{2} S_{020} + S_{110}$, $T_{020} = \frac{1}{2} S_{020}$, $T_{101} = S_{101}$, $T_{011} = -\frac{1}{2} S_{020} + S_{011}$, $T_{002} = S_{002}$. These coefficients now express the polynomial L with respect to the L-basis defined by the knot-net $\{\{L_{1j}\}, \{M_{2j}\}, \{M_{3j}\}, j = 1, 2\}$.

Finally we repeat the above procedure with a third tetrahedron, where the coefficients T_α are now placed at the nodes α with $|\alpha| = 2$ as shown in the rightmost diagram of Figure 5.2. The labels on the tetrahedron are now permuted from (i, j, k) to (j, k, i) because now we wish to retain the polynomials M_{2j} and M_{3j} and replace the polynomials L_{1j} by M_{1j}. Now the labels $g_{k,\alpha}$ are computed as follows: For $|\alpha| = 0, 1$, let $i = 2 - \alpha$; then

$$L_{1i} = g_{1,\alpha} M_{1,\alpha_1+1} + g_{2,\alpha} M_{2,\alpha_2+1} + g_{3,\alpha} M_{3,\alpha_3+1}.$$

Again in our special case these labels are the same as in the first tetrahedron and are shown in the right diagram of Figure 5.2. After the computation is complete, the new coefficients U_α emerge on the nodes at the base triangle. These new coefficients are as follows: $U_{200} = \frac{1}{2}T_{200}$, $U_{110} = -\frac{1}{2}T_{200} + T_{110}$, $U_{020} = T_{020}$, $U_{101} = -\frac{1}{2}T_{200} + T_{101}$, $U_{011} = T_{011}$, $U_{002} = T_{002}$. These coefficients express the polynomial L with respect to the L-basis defined by the knot-net $\mathcal{M} = \{\{M_{1j}\}, \{M_{2j}\}, \{M_{3j}\}, j = 1, 2\}$. The change of basis algorithm is now complete. The final coefficients T_α, are: $U_{200} = \frac{1}{2}R_{200}$, $U_{110} = -\frac{1}{2}R_{200} - \frac{1}{2}R_{020} + R_{110}$, $U_{020} = \frac{1}{2}R_{020}$, $U_{101} = -\frac{1}{2}R_{200} - \frac{1}{2}R_{002} + R_{101}$, $U_{011} = -\frac{1}{2}R_{020} - \frac{1}{2}R_{002} + R_{011}$, $U_{002} = \frac{1}{2}T_{002}$.

The general change of basis algorithm from any L-basis to any other L-basis is obtained by following essentially the same procedure. Suppose we are given the coefficients R_α of a polynomial L of degree n with respect to an L-basis $\{l^n_\alpha\}$ defined by the knot-net $\mathcal{L} = \{\{L_{1j}\}, \{L_{2j}\}, \{L_{3j}\}, j = 1, \cdots, n\}$. We would like to compute the coefficients U_α of this polynomial L with respect to another L-basis $\{p^n_\alpha\}$ defined by another knot-net $\mathcal{M} = \{\{M_{1j}\}, \{M_{2j}\}, \{M_{3j}\}, j = 1, \cdots, n\}$.

The general change of basis algorithm is constructed in the following manner:

1. Build three tetrahedra. For each tetrahedron, $\frac{(n+1-i)(n+2-i)}{2}$ nodes are placed at the i-th level of the tetrahedron for $i = 0, \cdots, n$. The labels $g_{k,\alpha}$ along the edges of the first tetrahedron are computed for $|\alpha| = 0, \cdots, n-1$, from

$$L_{3i} = g_{1,\alpha}L_{1,\alpha_1+1} + g_{2,\alpha}L_{2,\alpha_2+1} + g_{3,\alpha}M_{3,\alpha_3+1}, \quad i = n - |\alpha|.$$

   The labels for the second and the third tetrahedron are computed in a similar fashion. We assume that the intermediate knot-nets $\{\{L_{1j}\}, \{L_{2j}\}, \{M_{3j}\}, j = 1, \cdots, n\}$ are linearly independent.

2. Point the arrows on the tetrahedron downwards and place the original coefficients R_α along the lateral face of the pyramid. Carry out the computation and collect the new coefficients S_α along the base of the pyramid.

3. Repeat steps 1 and 2 twice with the second and third tetrahedron using the output of the previous step as the input of the next step. After 3 steps, the coefficients at the base of the tetrahedron are the desired coefficients U_α.

We can use this general change of basis algorithm for L-bases to convert from Bézier to Lagrange form. Since the Lagrange coefficients are the values of the bivariate polynomial at $O(n^2)$ nodes and since this change of basis algorithm is $O(n^3)$, converting from Bézier to Lagrange form evaluates the polynomial at $O(n^2)$ points with an amortized cost of $O(n)$ computations per point. This cost compares very favorably with the de Casteljau evaluation algorithm for Bézier surfaces which costs $O(n^3)$ computations per point.

Finally, the transformation between a B-basis and an L-basis can be achieved by factoring through the Bézier or multinomial bases, which are both B-bases and L-bases. For example, given a polynomial with respect to a power B-basis one can convert from the power B-basis to either a multinomial or Bézier basis using the change of basis algorithms between B-bases [LG95] and then convert from the multinomial or Bézier basis to the desired L-basis, say a Lagrange L-basis, by using the change of basis algorithms between L-bases described above. Again when the L-basis is a Lagrange basis, this change of basis algorithm evaluates the B-patch at $O(n^2)$ points with an amortized cost of $O(n)$ computations per point. This compares favorably with the generalized de Boor evaluation algorithm for B-patches which requires $O(n^3)$ computations per point.

6. Conclusions and Future Work

Lagrange and Newton bases play a very important role in point and derivative interpolation problems for surfaces. We have demonstrated that a very interesting subclass of bivariate Lagrange bases can be realized as bivariate L-bases. We have also demonstrated that a very interesting subclass of bivariate Newton bases can be realized as bivariate L-bases. Using the principle of duality between L-bases and B-bases, we have established that Lagrange L-bases and generalized Newton L-bases are dual respectively to power B-bases and generalized Newton dual B-bases respectively. We have also discussed the duality between Bézier and multinomial bases, which arise as special cases of both B-bases and L-bases. We went on to provide a geometric interpretation of the duality principle as point-line duality, where a point or a vector in a B-basis corresponds to a line in an L-basis. This duality provides strong geometric insight for working with these bases.

We have presented a unified collection of change of basis algorithms based on the principle of duality for a wide variety of polynomial bases used in representing surfaces including the Bézier, multinomial, Lagrange, power, Newton, Newton dual, B-bases, and L-bases. We have also given an example of change of basis from Lagrange to Bézier basis.

This research has opened up several interesting new questions. The generalization of the de Boor evaluation algorithm for bivariate B-bases is very well-known. The dual evaluation algorithm for bivariate L-bases was described by the authors in an earlier work [LG95]. Does this algorithm yield new algorithms for the evaluation of multinomial, Lagrange and Newton bases? The de Casteljau subdivision algorithm for a Bézier patch is very well-known. What is the corresponding dual algorithm? We plan to investigate the dual evaluation algorithms for L-patches [LG94c], dual de Casteljau subdivision algorithm for Bézier surfaces [LG94a] and duality between degree elevation and differentiation formulas [LG94b] in forthcoming papers. Although we have discussed point-line duality, we have observed that this duality is not self-dual. It will be interesting to explore self-duality and discover new computational algorithms based on self-duality. It would also be worthwhile to extend the notions of B-bases and L-bases to enlarge the configurations of points and lines for which Lagrange or Newton bases exist but for which a Lagrange L-basis or Newton L-basis does not exist.

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6.1 Appendix

Let $\{\ \mathbf{u}_{1j},\ \mathbf{u}_{2j},\ \mathbf{u}_{3j},\ j = 1, \cdots, n\ \}$ be a knot-net of vectors. Suppose the vectors $(\mathbf{u}_{1,\alpha_1+1}, \mathbf{u}_{2,\alpha_2+1}, \mathbf{u}_{3,\alpha_3+1})$ are linearly dependent for $|\alpha| = n$, $0 \le \alpha_k \le n-1$. This appendix gives an inductive proof of the fact that up to constant multiples each element of the corresponding B-basis b_α^n is an n-th power of a linear polynomial. The inductive proof is very interesting in its own right and reveals the underlying structure of the recurrence diagram. Also the technique used in the proof is valuable in other situations, including the proof of the generalized Aitken-Neville algorithm for Lagrange L-bases [LG94c].

Let q_α be the equations of the lines associated with the power B-basis as defined in Section 4.2.2.
Theorem 2:

$$b_\alpha^n(\mathbf{u}) = \frac{(q_\alpha)^n}{\prod_{j=1,\cdots,\alpha_i;i=1,2,3} q_\alpha(\mathbf{u}_{ij})}.$$

Proof: The proof is by induction. The case $n = 1$ reduces to the simple case of a triangle defined by the points $\mathbf{u}_{11}$, $\mathbf{u}_{21}$ and $\mathbf{u}_{31}$. The lines opposite to these vertices are Q_{100}, Q_{010} and Q_{001} respectively. In this case it can readily be verified that the basis functions b_{100}^1, b_{010}^1 and b_{001}^1 are $\frac{q_{100}}{q_{100}(\mathbf{u}_{11})}$, $\frac{q_{010}}{q_{010}(\mathbf{u}_{21})}$ and $\frac{q_{001}}{q_{001}(\mathbf{u}_{31})}$ respectively. If the three points lie in the affine plane, then these basis functions are simply the barycentric coordinates with respect to the triangle formed by $\mathbf{u}_{11}$, $\mathbf{u}_{21}$ and $\mathbf{u}_{31}$.

The inductive hypothesis assumes that the statement of the theorem is true for $n-1$. We now prove that the statement holds for n. To this purpose, choose an arbitrary but fixed α. Recall that b_α^n is defined from the recurrence in Equation 2.2 by substituting $C_\alpha = 1$ and setting the other constants $C_\beta = 0$. With this choice of constants, at the highest level of the recurrence when $|\alpha| = 0$, we make the following observations:

1. $C_0^n(\mathbf{u}) = b_\alpha^n$.
2. $h_{k,000}(\mathbf{u})$ are the coordinates of $\mathbf{u}$ with respect to the triangle $(\mathbf{u}_{11}, \mathbf{u}_{21}, \mathbf{u}_{31})$, that is, $h_{1,000}(\mathbf{u}) = \frac{q_{100}(\mathbf{u})}{q_{100}(\mathbf{u}_{11})}$, $h_{2,000}(\mathbf{u}) = \frac{q_{010}(\mathbf{u})}{q_{010}(\mathbf{u}_{21})}$, and $h_{3,000}(\mathbf{u}) = \frac{q_{001}(\mathbf{u})}{q_{001}(\mathbf{u}_{31})}$.
3. Finally, $C_{e_k}^{n-1}(\mathbf{u})$ for $k = 1,2,3$ are obtained by running only $n-1$ levels of the recurrence in Equation 2.2, and therefore, $C_{e_k}^{n-1}(\mathbf{u}) = b_{\alpha-e_k}^{n-1}(\mathbf{u})$, where $b_{\alpha-e_k}^{n-1}(\mathbf{u})$ are the B-basis functions corresponding to the knot-nets $\mathcal{W}_1 = \{(\hat{\mathbf{u}}_{11}, \cdots, \mathbf{u}_{1n}), (\mathbf{u}_{21}, \cdots, \hat{\mathbf{u}}_{2n}), (\mathbf{u}_{31}, \cdots, \hat{\mathbf{u}}_{3n})\}$, $\mathcal{W}_2 = \{(\mathbf{u}_{11}, \cdots, \hat{\mathbf{u}}_{1n}), (\hat{\mathbf{u}}_{21}, \cdots, \mathbf{u}_{2n}), (\mathbf{u}_{31}, \cdots, \hat{\mathbf{u}}_{3n})\}$, and $\mathcal{W}_3 = \{(\mathbf{u}_{11}, \cdots, \hat{\mathbf{u}}_{1n}), (\mathbf{u}_{21}, \cdots, \hat{\mathbf{u}}_{2n}), (\hat{\mathbf{u}}_{31}, \cdots, \mathbf{u}_{3n})\}$ respectively, where $\hat{\mathbf{u}}$ means that the term $\mathbf{u}$ is missing.

Putting these observations together, Equation 2.2 at the highest level of recurrence when $|\alpha| = 0$ now translates into the following:

$$b_\alpha^n(\mathbf{u}) = \frac{q_{100}(\mathbf{u})}{q_{100}(\mathbf{u}_{11})} b_{\alpha-e_1}^{n-1}(\mathbf{u}) + \frac{q_{010}(\mathbf{u})}{q_{010}(\mathbf{u}_{21})} b_{\alpha-e_2}^{n-1}(\mathbf{u}) + \frac{q_{001}(\mathbf{u})}{q_{001}(\mathbf{u}_{31})} b_{\alpha-e_3}^{n-1}(\mathbf{u}). \tag{6.1}$$

We now prove that the knot-nets $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$ satisfy the linear independence condition so that they actually are knot-nets and that they also satisfy the linear dependence condition of the power basis.

To this purpose, let us denote the knot-net of $\mathcal{W}_1$ also as follows: $\{\mathbf{w}_{1j}, \mathbf{w}_{2j}, \mathbf{w}_{3j}, j = 1, \cdots, 3\}$. The knot-net $\mathcal{W}_1$ satisfies the linear independence condition because $(\mathbf{w}_{1,\beta_1+1}, \mathbf{w}_{2,\beta_2+1}, \mathbf{w}_{3,\beta_3+1})$

is linearly independent for $0 \le |\beta| \le n-2$ iff $(\mathbf{u}_{1,\beta_1+2}, \mathbf{u}_{2,\beta_2+1}, \mathbf{u}_{3,\beta_3+1})$ is linearly independent for $0 \le |\beta| \le n-2$. The latter condition is, however, equivalent to the linear independence of $(\mathbf{u}_{1,\alpha_1+1}, \mathbf{u}_{2,\alpha_2+1}, \mathbf{u}_{3,\alpha_3+1})$ where $\alpha = (\beta_1+1, \beta_2, \beta_3)$ with $1 \le |\alpha| \le n-1$, which is satisfied because of the linear independence condition on the original knot-net $\mathcal{U}$. Similarly, the knot-nets $\mathcal{W}_2$ and $\mathcal{W}_3$ are linearly independent.

Moreover, the knot-net $\mathcal{W}_1$ satisfies the linear dependence condition of the power basis because $(\mathbf{w}_{1,\beta_1+1}, \mathbf{w}_{2,\beta_2+1}, \mathbf{w}_{3,\beta_3+1})$ are linearly dependent for $|\beta| = n-1$, $0 \le \beta_k \le n-2$ iff $(\mathbf{u}_{1,\beta_1+2}, \mathbf{u}_{2,\beta_2+1}, \mathbf{u}_{3,\beta_3+1})$ are linearly dependent for $|\beta| = n-1$, $0 \le \beta_k \le n-2$. The latter condition is, however, equivalent to the linear dependence of $(\mathbf{u}_{1,\alpha_1+1}, \mathbf{u}_{2,\alpha_2+1}, \mathbf{u}_{3,\alpha_3+1})$ where $\alpha = (\beta_1+1, \beta_2, \beta_3)$ with $|\alpha| = n$, $0 \le \alpha_k \le n-1$, which is satisfied because of the linear dependence condition on the original knot-net $\mathcal{U}$. Similarly, the knot-nets $\mathcal{W}_2$ and $\mathcal{W}_3$ satisfy the linear dependence condition of the power basis.

Since the knot-nets $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$ satisfy the assumptions of the theorem, we can apply the inductive hypothesis to these knot-nets. Now observe that the line corresponding to the power B-basis $b^{n-1}_{\alpha-e_1}$ with $|\alpha| = n$, is the line determined by $\mathbf{w}_{1,\alpha_1}$, $\mathbf{w}_{2,\alpha_2+1}$ and $\mathbf{w}_{3,\alpha_3+1}$. which in turn is the line determined by $(\mathbf{u}_{1,\alpha_1+1}, \mathbf{u}_{1,\alpha_2+1}, \mathbf{u}_{3,\alpha_3+1})$, and is therefore q_α. Similar assertions hold for $b^{n-1}_{\alpha-e_2}$ and $b^{n-1}_{\alpha-e_3}$. Hence the inductive hypothesis yields:

$$b^{n-1}_{\alpha-e_k}(\mathbf{u}) = \frac{q_\alpha(\mathbf{u}_{k1})}{\prod_{j=1,\cdots,\alpha_i;i=1,2,3} q_\alpha(\mathbf{u}_{ij})} \left(q_\alpha(\mathbf{u})\right)^{n-1}.$$

Substituting this formula into Equation 6.1, we obtain:

$$b^n_\alpha(\mathbf{u}) = \frac{(q_\alpha(\mathbf{u}))^{n-1}}{\prod_{j=1,\cdots,\alpha_i;i=1,2,3} q_\alpha(\mathbf{u}_{ij})} \left(\frac{q_\alpha(\mathbf{u}_{11})}{q_{100}(\mathbf{u}_{11})} q_{100}(\mathbf{u}) + \frac{q_\alpha(\mathbf{u}_{21})}{q_{010}(\mathbf{u}_{21})} q_{010}(\mathbf{u}) + \frac{q_\alpha(\mathbf{u}_{31})}{q_{001}(\mathbf{u}_{31})} q_{001}(\mathbf{u}) \right). \tag{6.2}$$

Now observe that the expression I within the brackets in Equation 6.2 is a linear polynomial and is therefore completely determined by its value at three independent points. However, since $I(\mathbf{u}_{k1}) = q_\alpha(\mathbf{u}_{k1})$ for $k = 1, 2, 3$ and $\mathbf{u}_{11}$, $\mathbf{u}_{21}$, and $\mathbf{u}_{31}$ are linearly independent points, it follows that $I = q_\alpha(\mathbf{u})$. Thus the statement of the theorem is established. □