

Scattering Parameter Transient Analysis of Interconnect Networks with Nonlinear Terminations Using Recursive Convolution

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ABSTRACT

A novel method for analyzing interconnect networks with nonlinear terminations is presented. The circuit is partitioned into linear and nonlinear networks. A scattering parameter based macromodel is introduced to model the linear network. An efficient network reduction algorithm is developed to reduce the linear network into a network containing one multiport component (macromodel) together with sources and loads of interest. Exponentially Decayed Polynomial Functions(EDPF) are used to approximate the scattering parameters of the macromodel, which is always stable for stable circuits. In order to incorporate the macromodel into a SPICE like circuit simulator, a simplified recursive convolution formula is developed and Norton equivalent circuits are derived based on recursive convolution. Experiment results indicate that our method can approach the accuracy of SPICE3e2 with order of magnitudes less computing time.

Keywords: scattering parameter, macromodel, multiport component, multiport interconnect node, component merging, network reduction, lossy transmission line, exponentially decayed polynomial function, nonlinear loads, recursive convolution.

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Using Recursive Convolution

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1 Introduction

As the electrical length of interconnects becomes a significant fraction of signal wavelength during the fastest transient, the conventional lumped-impedance interconnect model becomes inadequate and distributed coupled transmission line effects must be taken into account for both on-chip and off-chip interconnects. However, the fundamental difficulty encountered in integrating transmission line simulation in a transient circuit simulator arises because the circuits containing nonlinear devices must be characterized in the time domain while transmission lines with loss, dispersion, and interconnect discontinuities are best modeled in the frequency domain. To cope with this difficulty, direct convolution techniques are used. The system outputs are the convolutions of the inputs with the impulse responses. The fundamental problem lies in how to determine the impulse response of an arbitrary interconnect system. While explicit analytical expression of the impulse responses is impractical, the numerical inverse Fast Fourier Transformation technique suffers from the fact that excessive number of frequency points are needed to avoid aliasing effects. The another drawback of the direct convolution is time consuming.

In order to deal with these difficulties, Pade technique[2, 16] is used to get an approximated explicit analytical expression of the transfer function. Impulse response functions are approximated with sum of exponential functions in time domain. To avoid folding, sliding, multiplication, and integration in the progressively time-consuming direct convolution integration process, the recursive formula[15] is rediscovered[16, 17] for computing convolution of the approximated impulse response function with any other function. However, Pade technique suffers from the unstable problem: unstable poles may be

generated for known stable networks. Instead of using Pade technique, based on the method of inversion of Laplace transform, F. Y. Chang[14] introduced Laguerre function to approximate the impulse response functions, together with a recursive convolution formula. But the accuracy of the approximation is very sensitive to the time constant which is chosen based on a very rough empirical formula. We have proposed an improved method [8] to choose the time constant by introducing an error function. In this paper, we simplify the recursive formula and derive Norton equivalent circuits based on the recursive formula.

The computation of impulse response of an interconnect system can be achieved by approximating the characteristic impedance and exponential wave propagation functions[14, 16], or by approximating state variables of state equations[17]. We have presented a scattering parameter based macromodel[7, 8] to compute transfer functions of interconnect systems. Scattering parameter (s-parameter) based methods provides efficient techniques to analyze practical analog and digital integrated circuit interconnect systems that can contain large number of coupled conductors and discontinuities[1, 9, 10]. S-parameters are well suited for the characterizing and modeling of linear high frequency devices, partially because it is easier to directly measure the scattering parameters of any components on broad frequency bands[13]. Alternatively, the port parameters of many passive components (such as transmission lines) can be determined in terms of their geometric dimensions and the electrical characteristics of the materials. S-parameters are numerically well defined and conceptually simple. The multiconductor transmission line segments and various discontinuities can be considered as separate components and their s-parameters can be obtained by measurement or software.

Transient analysis of interconnects based on s-parameters has been addressed by several authors recently[4, 5, 6]. But all of these methods are based on convolution which are very time consuming. A novel method for analyzing interconnect networks with nonlinear terminations is proposed in this paper. The method can handle general RLC and transmission line networks including capacitive or inductive cutsets and loops. The circuit are partitioned into linear and nonlinear networks. A scattering parameter based macromodel is introduced to model the linear network. An efficient network reduction algorithm is developed to reduce the original network into a network containing one multiport component (macromodel) together with sources and loads of interest, which may be nonlinear. Exponentially Decayed Polynomial Functions (EDPF)[14] are used to approximate the scattering parameters of the macromodel, which is always stable for stable circuits. The macromodel is very flexible that the accuracy of the model can be controlled by adjusting the order of approximation. The efficiency and accuracy of the macromodel are well demonstrated.

2 Scattering Parameter Based Macromodel

Given the individual component scattering parameters, we describe a systematic reduction algorithm to reduce a distributed-lumped network to a multiport with sources and loads of interest, as shown in Figure 1.

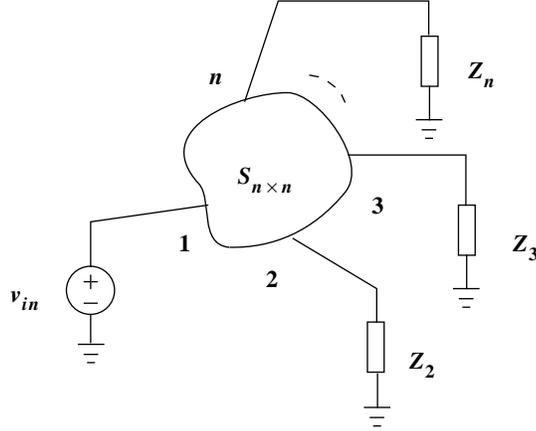


Figure 1. S-parameter based macromodel

The network reduction problem can be defined as follows[7, 8]: given a linear distributed-lumped network described by s-parameters, find a multiport representation of the network as illustrated by Figure 1, where the multiport is characterized by its s-parameters. All nodes in the network are internal to the multiport except the node connected to the driving source (node 1) and the loads of interest (nodes 2 through n). These external nodes are specified by the user.

To obtain such a multiport representation with m external ports from an arbitrary distributed-lumped network of n original nodes, the network is reduced by merging the nodes into the multiport one at a time while keeping all user specified nodes external. There are two basic reduction rules[7, 8]:

Adjoined Merging Rule: Let X and Y be two adjacent multiports, with m ports and n ports respectively. Assume port k of X is connected to port l of Y , as shown in Figure 2. After

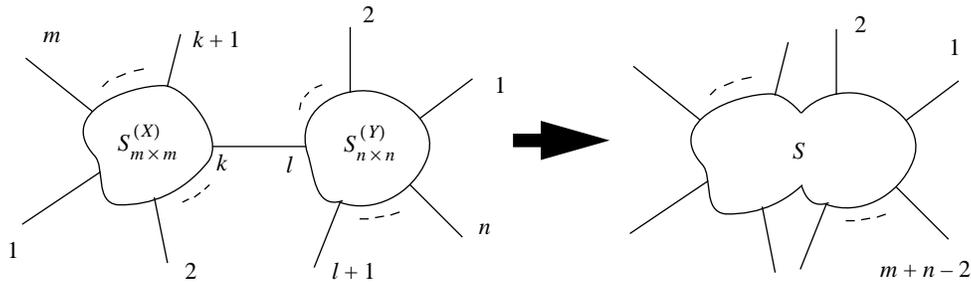


Figure 2. Adjoined Merging.

merging X and Y , the resultant $(n + m - 2)$ port has the following s-parameters:

$$S_{ji} = \begin{cases} S_{ji}^{(X)} + \frac{S_{ki}^{(X)} S_{ll}^{(Y)} S_{jk}^{(X)}}{1 - S_{kk}^{(X)} S_{ll}^{(Y)}} & i, j \in X \\ S_{ji}^{(Y)} + \frac{S_{li}^{(Y)} S_{kk}^{(X)} S_{jl}^{(Y)}}{1 - S_{kk}^{(X)} S_{ll}^{(Y)}} & i, j \in Y \\ \frac{S_{ki}^{(X)} S_{jl}^{(Y)}}{1 - S_{kk}^{(X)} S_{ll}^{(Y)}} & i \in X, j \in Y \end{cases} \quad (1)$$

Self Merging Rule: Let X be an m port with a self loop connected to the l^{th} and k^{th} ports, as shown in Figure 3. After eliminating the self loop, the resultant $(m - 2)$ port has the

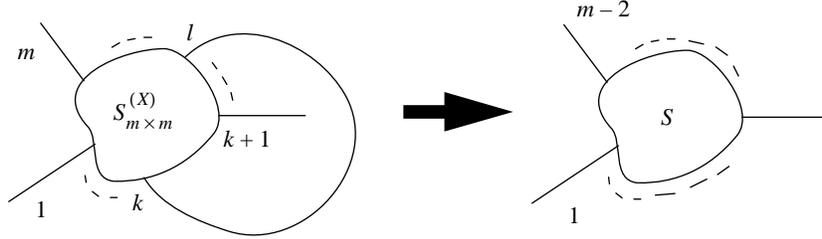


Figure 3. Self Merging.

following s-parameters:

$$S_{ji} = S_{ji}^{(X)} + S_{jl}^{(X)} a_l + S_{jk}^{(X)} a_k \quad i, j = 1, 2, \dots, m - 2 \quad (2)$$

where

$$\begin{aligned} a_l &= \frac{1}{\Delta} \left(S_{li}^{(X)} S_{kk}^{(X)} + S_{ki}^{(X)} \left(1 - S_{lk}^{(X)} \right) \right) \\ a_k &= \frac{1}{\Delta} \left(S_{ki}^{(X)} S_{ll}^{(X)} + S_{li}^{(X)} \left(1 - S_{kl}^{(X)} \right) \right) \\ \Delta &= \left(1 - S_{lk}^{(X)} \right) \left(1 - S_{kl}^{(X)} \right) - S_{ll}^{(X)} S_{kk}^{(X)} \end{aligned} \quad (3)$$

For an arbitrary distributed-lumped network described by the linear components, the Adjoined Merging Rule is used to merge all internal components, and the Self Merging rule is applied to eliminate the self loops introduced by the Adjoined Merging process. The macromodel, or the voltage transfer function of the network can be characterized by the s-parameters of the multiport component resulted from the reduction process, together with the s-parameters of the loads.

It should be pointed out that the above reduction process does not require the network be an RC tree. Since we start with the s-parameter description of the system, which always

exists for any physically realizable system. The formulation is completely general for any linear distributed-lumped network with scattering parameter descriptions. Another advantage is that the need for using lumped representation of transmission lines is eliminated since lossy transmission lines can be represented in a distributed form.

3 Time Domain Description of Macromodel with Exponentially Decayed Polynomial Function(EDPF)

In order to derive recursive convolution, the s-parameters of the macromodel should be fit with an explicit expression. Exponentially Decayed Polynomial Function (EDPF)[8, 14] is used here, since it can approximate the time domain transfer function with any degree of accuracy. And the corresponding frequency domain function of EDPF have only one repeated stable pole, so it is always stable for stable networks. An n-th order EDPF in time domain has the following form:

$$h(t) = \left(p_0 + p_1 \frac{t}{d} + p_2 \left(\frac{t}{d} \right)^2 + \dots + p_n \left(\frac{t}{d} \right)^n \right) e^{-(t/d)} \quad (4)$$

where d is the time constant which can be used to control the accuracy. We will give two efficient methods to model a frequency domain s-parameter $S(s)$ with EDPF.

3.1 Modeling of the S-parameters Using Least Square Method

In order to compute coefficients p_i ($i = 0, \dots, n$) of EDPF, we transfer $h(t)$ into frequency domain,

$$H(s) = \sum_{i=0}^n p_i H_i(s) = \sum_{i=0}^n p_i \frac{i! d}{(1+ds)^{i+1}} \quad (5)$$

where $H_i(s)$ are called basis functions. For this linear model, we define a merit function

$$\delta = \sum_{k=0}^m \left(S(s_k) - \sum_{i=0}^n p_i H_i(s_k) \right)^2 \quad (6)$$

where m is the number of frequency points to be fit, $S(s_k)$ is the s-parameter to be fit at frequency s_k . The least square fitting is used to find coefficients p_i which minimize δ . It could be proved [18] that the unknown coefficients $P = \{p_0, p_1, \dots, p_n\}$ can be determined by solving the following linear equation:

$$XP = Q \quad (7)$$

where X is the $(n+1) \times (n+1)$ matrix with elements

$$x_{ij} = \sum_{k=0}^m H_i(s_k) H_j(s_k) \quad i, j = 0, 1, \dots, n \quad (8)$$

and Q is the $n+1$ vector with elements

$$q_i = \sum_{k=0}^m H_i(s_k) S(s_k) \quad i = 0, 1, \dots, n \quad (9)$$

3.2 Modeling of the S-parameters Based on Numerical Inversion of Laplace Transform

In order to compute these coefficients p_i ($i = 0, \dots, n$), rewrite $h(t)$ in a series of orthogonal functions:

$$h(t) = \sum_{i=0}^n c_i L_i\left(\frac{t}{T}\right) e^{-(t/2T)} \quad (10)$$

where $T = d/2$, and $L_i(x)$ are Laguerre polynomials of x . Laguerre polynomials are widely used in the numerical inversion of Laplace transforms[22]. A polynomial function is the linear combination of set of Laguerre polynomials[14],

$$\begin{aligned} L_0\left(\frac{t}{T}\right) &= 1 \\ L_1\left(\frac{t}{T}\right) &= \frac{t}{T} - 1 \\ L_i\left(\frac{t}{T}\right) &= \frac{1}{i} \left\{ \left[\frac{t}{T} - (2i-1) \right] L_{i-1}\left(\frac{t}{T}\right) - (i-1) L_{i-2}\left(\frac{t}{T}\right) \right\} \quad (i \geq 2) \end{aligned} \quad (11)$$

The Laplace transform of $h(t)$ is

$$H(s) = \sum_{i=0}^n c_i \frac{(1/2T-s)^i}{(1/2T+s)^{i+1}} \quad (12)$$

The coefficients c_i can be found as follows. Rewrite above equation

$$\hat{H}(s) = (1/2T+s)H(s) = \sum_{i=0}^n c_i \left(\frac{1/2T-s}{1/2T+s} \right)^i \quad (13)$$

Let $e^{j\theta} = (1/2T-s)/(1/2T+s)$, thus

$$\omega = \frac{1}{2T} \cot \frac{\theta}{2} \quad (14)$$

and $\hat{H}(s)$ becomes

$$\hat{H}\left(j\frac{1}{2T} \cot \frac{\theta}{2}\right) = \sum_{i=0}^n c_i e^{jn\theta} \quad (15)$$

the right side of the above equation is a Fourier series. In order to use FFT technique to evaluate these coefficients, equally spaced values of θ should be chosen. The corresponding values of ω are determined by the equation (14).

4 Norton Equivalent Circuits Based on Recursive Convolution

4.1 Recursive Convolution Based on the EDPF

It has been shown that convolution integration for a EDPF and a piecewise linear function can be computed by a recursive formula[14]. In the following, we state a simplified recursive convolution formula. For the derivation of the formula, see Appendix A. Consider the following convolution integration

$$y(t) = h(t) * x(t) = \left(\sum_{i=0}^n p_i \left(\frac{t}{d} \right)^i e^{-t/d} \right) * x(t) = \sum_{i=0}^n p_i w_i(t) \quad (16)$$

where

$$w_i(t) = \int_0^t \left(\frac{t-\lambda}{d} \right)^i e^{-(t-\lambda)/d} x(\lambda) d\lambda \quad (17)$$

then $y(t)$ can be computed by

$$y(t + \Delta t) = \phi(t + \Delta t) + \rho(t + \Delta t) x(t + \Delta t) \quad (18)$$

where $\rho(t + \Delta t) = p_0 \Delta t / 2$ and $\phi(t + \Delta t)$ can be computed with the following recursive formulas:

$$\begin{aligned} \phi(t + \Delta t) &= \sum_{i=0}^n p_i \zeta_i(t + \Delta t) \\ \zeta_i(t + \Delta t) &= e^{-\Delta t/d} \left(\sum_{k=0}^i \binom{i}{k} \left(\frac{\Delta t}{d} \right)^{i-k} w_k(t) \right) + \frac{\Delta t}{2} \left(\frac{\Delta t}{d} \right)^i x(t) \\ w_k(t + \Delta t) &= \zeta_k(t + \Delta t) + \frac{\Delta t}{2} x(t + \Delta t) \end{aligned} \quad (19)$$

4.2 Norton Equivalent Circuits

In order to incorporate the macromodel described with $N \times N$ s-matrix, shown in Figure 1, into SPICE like simulators, we derive a Norton equivalent circuit. By modifying the approach in [5], we insert two series impedances Z_0 and $-Z_0$ at each port to remove the effect of the reference impedance. Thus a virtual node is created between the load and the macromodel as shown in Figure 4 where v_i is the node voltage in frequency domain at port i and v_i is referred to as the virtual voltage at port i . These variables are related by

$$V_i = V_i + Z_0 I_i \quad i = 1, \dots, N \quad (20)$$

where I_i is the current flowing into the macromodel at port i . From the definition of s-parameter and the relation between circuit parameters and wave parameters, we have

$$b_i = \sum_{j=1}^N S_{ij} a_j \quad (21)$$

$$V_i = a_i + b_i \quad Z_0 I_i = a_i - b_i \quad (22)$$

where S_{ij} is a element of the s-matrix of the multiport component. a_i and b_i are

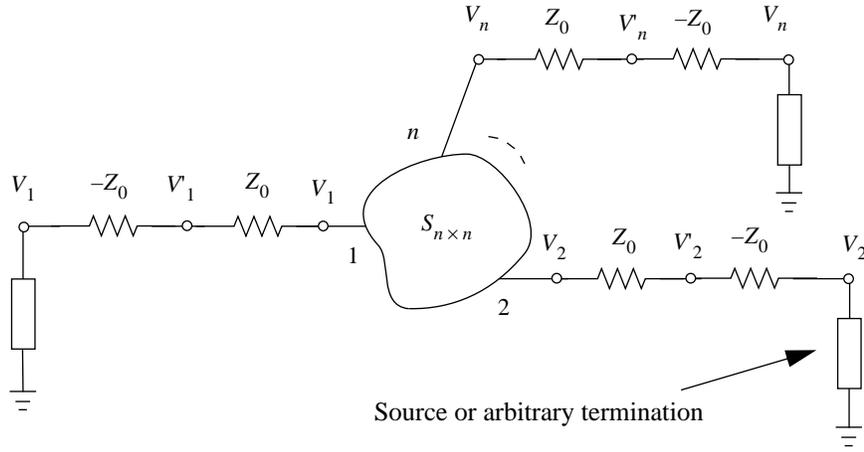


Figure 4. Macromodel with virtual nodes

incoming wave and outgoing wave at port i , respectively. From Eq. (20) and (22), we get

$$a_i = V_i/2 \quad (23)$$

Rewrite Eq. (20) and combine it with Eq. (21), (22) and (23), the macromodel system can be described by the following equation:

$$\begin{aligned} I_i &= Z_0^{-1} v_i - Z_0^{-1} V_i \\ &= Z_0^{-1} v_i - Z_0^{-1} (a_i + b_i) \\ &= Z_0^{-1} v_i - Z_0^{-1} \left(a_i + \sum_{j=1}^N S_{ij} a_j \right) \\ &= \frac{Z_0^{-1}}{2} v_i - \frac{Z_0^{-1}}{2} \sum_{j=1}^N S_{ij} V_j \end{aligned} \quad (24)$$

Writing this equation in time domain leads to the convolution equation

$$i_i(t) = \frac{Z_0^{-1}}{2} v_i'(t) - \frac{Z_0^{-1}}{2} \sum_{j=1}^N S_{ij} * v_j'(t) \quad (25)$$

Since s-parameters are modeled with EDPF, using the recursive convolution formula Eq. (18) and (19), Eq. (25) can be written with

$$i_i(t) = \frac{Z_0^{-1}}{2} v_i'(t) - \frac{Z_0^{-1}}{2} \sum_{j=1}^N (\phi_j(t) + \rho_j(t) v_j'(t)) = \sum_{j=1}^N g_{ij}(t) v_j'(t) - I_{si}(t) \quad (26)$$

where

$$I_{si}(t) = \frac{Z_0^{-1}}{2} \sum_{j=1}^N \phi_j(t) \quad (27)$$

$$(28)$$

$$g_{ij}(t) = \begin{cases} Z_0^{-1} (1 - \rho_j(t)) / 2 & i = j \\ -Z_0^{-1} \rho_j(t) / 2 & i \neq j \end{cases} \quad (29)$$

A Norton equivalent circuit can be derived from Eq. (26) and it can be easily

integrated into SPICE-like simulators. The equivalent circuit of a two port macromodel is shown in Fig 5.

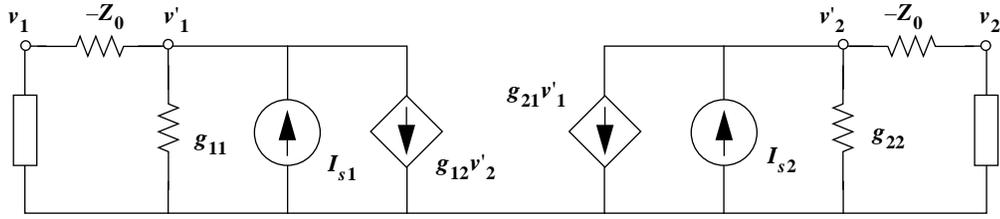


Figure 5. Equivalent circuit of a two port macromodel

5 Experimental Results

The model has been built and added to the Model Independent Simulator (MISIM) [19] which is based on the Modified Nodal Analysis (MNA). Figure 6 is a lossy transmission

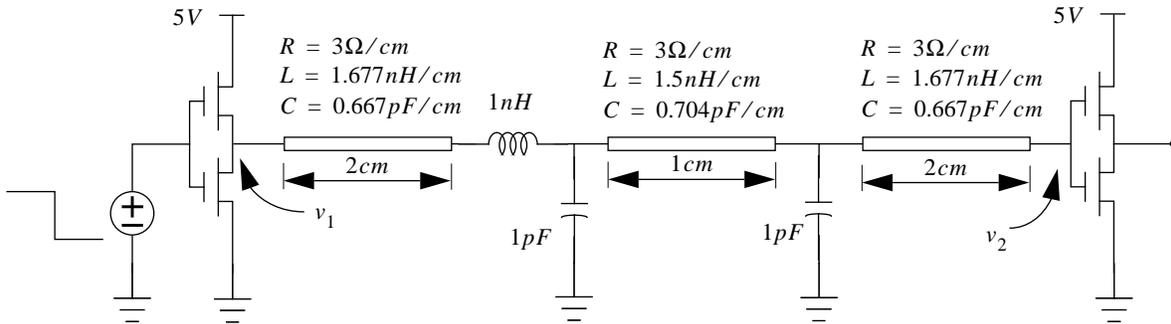


Figure 6. A lossy transmission line circuit with nonlinear loads

line circuit driven by a CMOS inverter. Figure 7a and Figure 7b are the response of the near end and the far end respectively. Comparing the direct convolution method (solid line)[20], our macromodel (dashed line) with recursive convolution has almost identical results with order of magnitudes less computing time.

The next example is a grid-type clock network (See Figure 8) which is distributed around the periphery of a $20mm \times 20mm$ chip. The vertical runs are on metal 1 ($R = 0.4\Omega/mm$, $L = 0.33nH/mm$ and $C = 0.434pF/mm$) and the horizontal runs are on metal 2 ($R = 0.5\Omega/mm$, $L = 0.413nH/mm$ and $C = 0.347pF/mm$). The network is represented by distributed lossy transmission lines driven by a CMOS inverter. Figure 9a is the curve of the driver output v_1 and Figure 9b is the curves of v_o . There is little difference between the results based on our macromodel and the SPICE model. While it took more than 500 CPU seconds on SUN SPARC 1+ for SPICE3e2 to get the answer using the direct convolution [21], our program

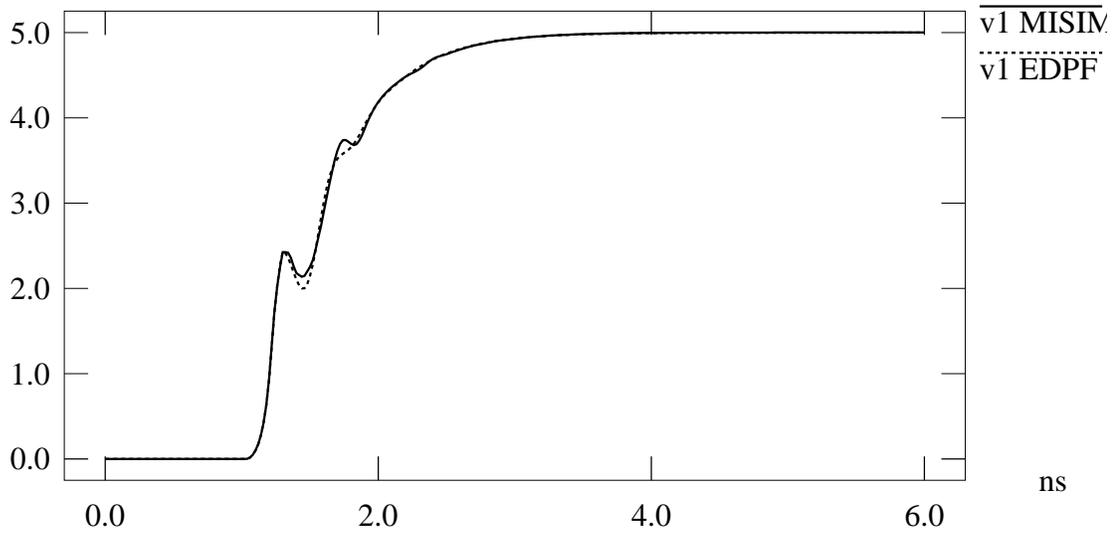


Figure 7a. Near end response of the lossy transmission line circuit

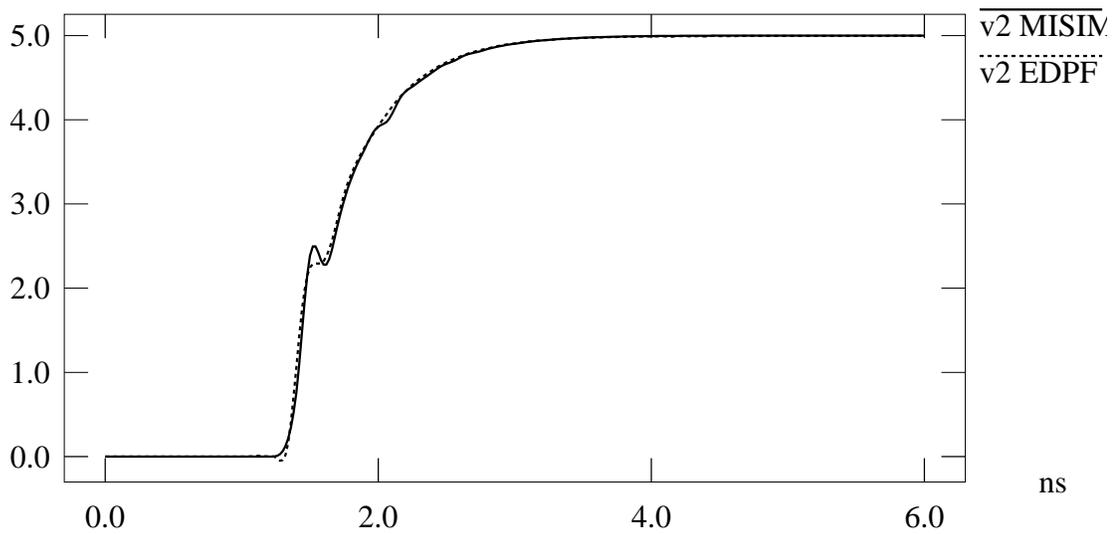


Figure 7b. Far end response of the lossy transmission line circuit took 56.3 seconds to analyze the circuit using 12th order EDPF approximation.

6 Conclusions

An efficient method for modeling arbitrary interconnect networks with linear or nonlinear loads for transient simulation was presented. The method is based on scattering parameter technique, a large scale interconnect system can be reduced to a network containing one multiport component (macromodel) together with sources, loads of interest and nonlinear elements. The scattering parameters of the model can be accurately fitted with Exponentially Decayed Polynomial Functions (EDPF) based on which a recursive convolution is derived. From the recursive convolution, a Norton equivalent circuit of the macromodel was derived.

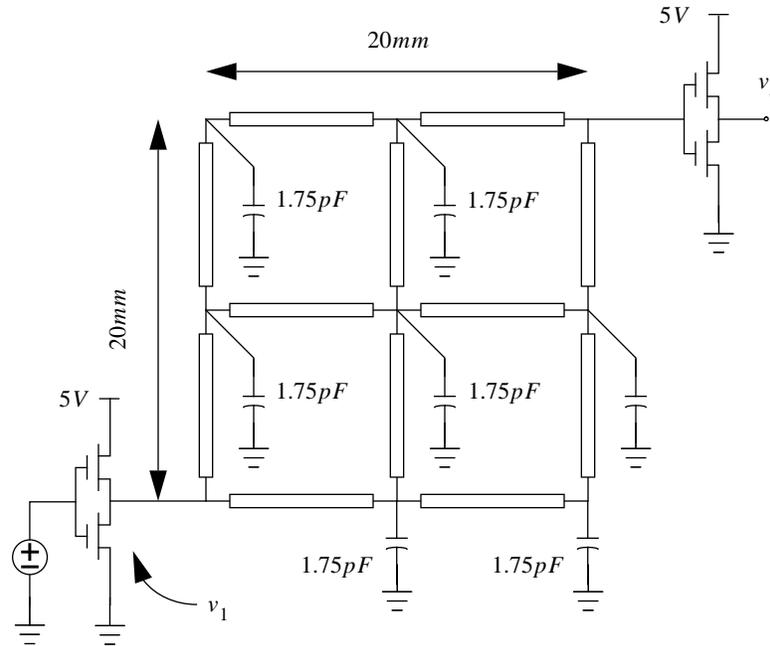


Figure 8. Grid-type clock network

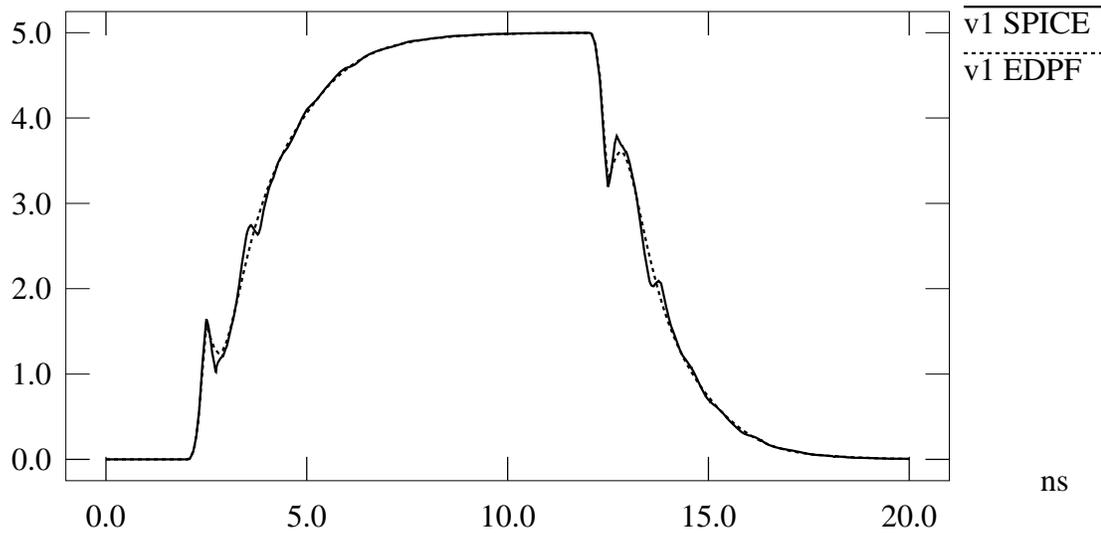


Figure 9a. Near end response of the clock network

The model was implemented in a transient simulator and it was verified with experiment results.

References

- [1] J. Dobrowolski, Introduction to Computer Methods for Microwave Circuit Analysis and Design, Artech House, 1991.
- [2] L. T. Pillage, and R. A. Rohrer, "Asymptotic Waveform Evaluation for Timing Analysis," *IEEE Trans. on CAD*, vol. CAD-9, April 1990.

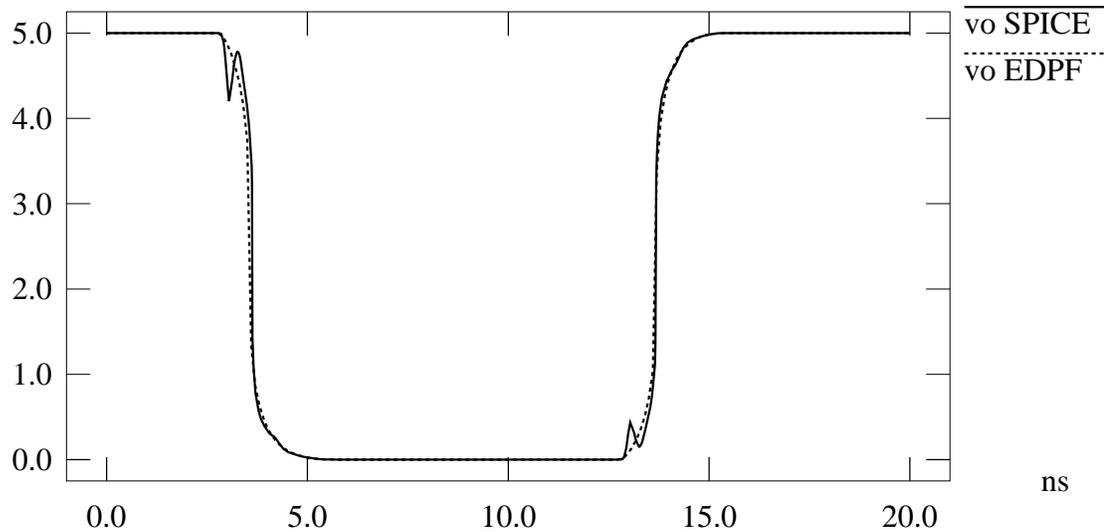


Figure 9b. Near end response of the clock network

- [3] L. T. Pillage, X. Huang, and R. A. Rohrer, "Asymptotic Waveform Evaluation for Circuits Containing Floating Nodes," *IEEE International Symposium on Circuits and Systems*, pp.613-616, May 1990.
- [4] J. E. Schutt-Aine, and R. Mittra, "Scattering Parameter Transient Analysis of Transmission Lines Loaded with Nonlinear Terminations," *IEEE Trans. on Microwave Theory Tech.*, vol. MTT-36, pp. 529-535, March 1988.
- [5] D. Winklestein, M. B. Steer, and R. Pomerleau, "Simulation of Arbitrary Transmission Line Networks with Nonlinear Terminations," *IEEE Trans. on Circuits and Systems*, vol. CAS-38, pp. 418-422, April 1991.
- [6] T. Dhaene, L. Martens, and D. De Zutter, "Transient Simulation of Arbitrary Nonuniform Interconnection Structures Characterized by Scattering Parameters," *IEEE Trans. on Circuits and Systems*, vol. CAS-39, pp. 928-937, Nov. 1992.
- [7] H. Liao, R. Wang, R. Chandra, and W. Dai, "S-Parameter Based Macro Model of Distributed-Lumped Networks Using Pade Approximation," *IEEE International Symposium on Circuits and Systems*, pp. 2319-2322, May 1993.
- [8] H. Liao, W. Dai, R. Wang, and F. Y. Chang, "S-Parameter Based Macro Model of Distributed-Lumped Networks Using Exponentially Decayed Polynomial Function," *Proceedings of 30th ACM/IEEE Design Automation Conference*, pp. 726-731, June 1993.
- [9] P. A. Rizzi, *Microwave Engineering*, Englewood Cliffs, NJ. Prentice-Hall, 1988.
- [10] K. C. Gupta, R. Grag, and R. Chadha, *Computer Aided Design of Microwave Circuits*, Dedham, MA: Artech 1981.
- [11] V. A. Monaco, and P. Tiberio, "Automatic Scattering Matrix Computation of Microwave Circuits," *Alta Frequenza*, vol. 39, pp. 59-64, 1970
- [12] N. Kuhn, "Simplified Signal Flow Graph Analysis," *Microwave J.*, VI, No. 11, pp. 59-66, 1963.
- [13] Hewlett-Packard. Application Note 117-1, 1969.
- [14] F. Y. Chang, "Transient Simulation of Nonuniform Coupled Lossy Transmission Lines Characterized with Frequency-Dependent Parameters, Part II: Discrete-Time Analysis," *IEEE Trans. on Circuits and Systems*, vol. CAS-39, pp. 907-927, Nov. 992.

- [15] A. Semlyen, and A. Dabuleanu, "Fast and Accurate Switching Transient Calculation on Transmission Lines with Ground Return Using Recursive Convolution," *IEEE Trans. on Power Apparatus and Systems*, vol. PAS-94, pp. 561-571, 1975.
- [16] S. Lin, and E. S. Kuh, "Transient Simulation of Lossy Interconnects Based on the Recursive Convolution Formulation," *IEEE Trans. on Circuits and Systems*, vol. CAS-39, pp. 879-892, Nov. 1992.
- [17] V. Raghavan, J. E. Bracken, and R. A. Rohrer, "AWESpice: A General Tool for the Efficient and Accurate Simulation of Interconnect Problems," *Proceedings of 29th ACM/IEEE Design Automation Conference*, pp. 81-86, June 1992.
- [18] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, Numerical Recipes in C: The Art of Scientific Computing, Cambridge, pp. 293, June 1991.
- [19] A. T. Yang, J. Yao, et al, User's Manual of MISIM, 1992.
- [20] D. S. Gao, A. T. Yang, and S. M. Kang, "Accurate Modeling and Simulation of Parallel Interconnection in High-speed Integrated circuits," *IEEE Trans. on Circuits and Systems*, pp. 1-9, Jan. 1992.
- [21] J. S. Roychowdury, and D. O. Pederson, "Efficient Transient Simulation of Lossy Interconnect," *Proceedings of 29th ACM/IEEE Design Automation Conference*, pp. 740-745, 1992.
- [22] W. T. Weeks, "Numerical Inversion of Laplace Transforms Using Laguerre Functions," *Journal of the Association for Computing Machinery*, vol. 13, pp. 419-426, July 1966.

Appendix A: Recursive Convolution Based on EDPF

Consider the following convolution

$$y(t) = h(t) * x(t) = \int_0^t h(t-\lambda) x(\lambda) d\lambda \quad (\text{a1})$$

where $x(t)$ is the excitation function, and $h(t)$ is the transfer function approximated in the exponentially decayed polynomial function (EDPF), that is

$$h(t) = \sum_{i=0}^n p_i \left(\frac{t}{d}\right)^i e^{-t/d} \quad (\text{a2})$$

Let

$$w_i(t) = \int_0^t \left(\frac{t-\lambda}{d}\right)^i e^{-(t-\lambda)/d} x(\lambda) d\lambda \quad (\text{a3})$$

then, equation (a1) has the form:

$$y(t) = \left(\sum_{i=0}^n p_i \left(\frac{t}{d}\right)^i e^{-t/d} \right) * x(t) = \sum_{i=0}^n p_i w_i(t) \quad (\text{a4})$$

Thus, the key point to efficiently compute $y(t)$ is how to update $g_i(t)$ at time $t + \Delta t$. Let us look at the recursive convolution of equation (a3) at time $t + \Delta t$

$$\begin{aligned} w_i(t + \Delta t) &= \int_0^{(t+\Delta t)} \left(\frac{t+\Delta t-\lambda}{d}\right)^i e^{-(t+\Delta t-\lambda)/d} x(\lambda) d\lambda \\ &= p_i(t + \Delta t) + q_i(t + \Delta t) \end{aligned} \quad (\text{a5})$$

where

$$\begin{aligned} p_i(t + \Delta t) &= \int_0^t \left(\frac{(t+\Delta t)-\lambda}{d}\right)^i e^{-(t+\Delta t-\lambda)/d} x(\lambda) d\lambda \\ &= e^{-\Delta t/d} \int_0^t \left(\frac{\Delta t+t-\lambda}{d}\right)^i e^{-(t-\lambda)/d} x(\lambda) d\lambda \\ &= e^{-\Delta t/d} \int_0^t \left(\sum_{k=0}^i \binom{i}{k} \left(\frac{\Delta t}{d}\right)^{i-k} \left(\frac{t-\lambda}{d}\right)^k \right) e^{-(t-\lambda)/d} x(\lambda) d\lambda \\ &= e^{-\Delta t/d} \sum_{k=0}^i \left(\binom{i}{k} \left(\frac{\Delta t}{d}\right)^{i-k} \int_0^t \left(\frac{t-\lambda}{d}\right)^k e^{-(t-\lambda)/d} x(\lambda) d\lambda \right) \\ &= e^{-\Delta t/d} \sum_{k=0}^i \left(\binom{i}{k} \left(\frac{\Delta t}{d}\right)^{i-k} w_k(t) \right) \end{aligned} \quad (\text{a6})$$

and

$$q_i(t + \Delta t) = \int_t^{(t+\Delta t)} \left(\frac{(t+\Delta t)-\lambda}{d}\right)^i e^{-(t+\Delta t-\lambda)/d} x(\lambda) d\lambda \quad (\text{a7})$$

By using the Trapezoidal rule for the above integration, we have

$$q_i(t + \Delta t) = \begin{cases} \frac{\Delta t}{2} \left(e^{-\Delta t/d} x(t) + x(t + \Delta t) \right) & i = 0 \\ \frac{\Delta t}{2} \left(\frac{\Delta t}{d} \right)^i e^{-\Delta t/d} x(t) & i > 0 \end{cases} \quad (\text{a8})$$

Now, combine equations (a4 – a7), we have the following recursive convolution

$$y(t + \Delta t) = \phi(t + \Delta t) + \rho(t + \Delta t) x(t + \Delta t) \quad (\text{a9})$$

where $\rho(t + \Delta t) = p_0 \Delta t / 2$ and $\phi(t + \Delta t)$ can be computed with the following recursive formulas:

$$\begin{aligned} \phi(t + \Delta t) &= \sum_{i=0}^n p_i \zeta_i(t + \Delta t) \\ \zeta_i(t + \Delta t) &= e^{-\Delta t/d} \left(\sum_{k=0}^i \left(\binom{i}{k} \left(\frac{\Delta t}{d} \right)^{i-k} w_k(t) \right) + \frac{\Delta t}{2} \left(\frac{\Delta t}{d} \right)^i x(t) \right) \\ w_k(t + \Delta t) &= \zeta_k(t + \Delta t) + \frac{\Delta t}{2} x(t + \Delta t) \end{aligned} \quad (\text{a10})$$