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shared by all NP problems, in particular their asymptotic probabilities obey a 0-1 law ([KV87]). On the other hand, the closure of strict Σ^1_1 formulae under polynomial reductions is the entire class of NP problems.

6 Concluding Remarks and Open Problems

In this paper we investigated NP optimization problems from the standpoint of logical definability and analyzed the relative expressive power of the various classes of NP optimization problems that arise in this framework. One of our findings is that logical definability has different implications for NP maximization problems than it has for NP minimization problems. The original motivation in [PY88] for pursuing the logical definability approach was to find syntactic classes of NP maximization problems with good approximation properties, such as MAX Σ_1, and to pinpoint natural complete problems for these classes. Since the class MIN Σ_1 contains non-approximable problems (modulo P$\neq$NP), it would be interesting to find syntactic subclasses of MIN Σ_1 that contain approximable problems only. Theorem 7 shows that the class MIN FΠ_1 is a first step in this direction.

The TRAVELING SALESMAN problem with distance $1, 2$ is an important example of a minimization problem that is approximable, but is not known to have a polynomial time approximation scheme. Papadimitriou and Yannakakis [PY90] have recently shown that every problem in the class MAX Σ_0 is L-reducible to the TRAVELING SALESMAN problem with distance 1,2. It is an open problem to identify a natural class of minimization problems for which the TRAVELING SALESMAN problem with distances $1, 2$ is complete.

Papadimitriou and Yannakakis [PY88] proved that MAX 3SAT and a host of other problems are complete for MAX Σ_0. Panconesi and Ranjan [PR90] introduced the problem MAX Number of Satisfiable Formulae (MAX NSF) and proved it complete for MAX Π_1. As mentioned earlier, it can be shown that this problem is also complete for the class MAX Π_2 = MAX $\mathcal{PB}$. It is not known, however, if MAX Σ_1 possesses complete problems. On the side of minimization, we proved here that MIN 3NT is complete for the class MIN Σ_0. It would be interesting to investigate the existence of complete problems for the classes MIN Σ_1 and MIN Π_1.

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Notice that $\bigwedge_i \psi_i$ is a CNF formula whose variables are of the form $S(\overline{\mathbf{y}})$, where $\overline{\mathbf{y}}$ is a sequence of length m. Without loss of generality, we can assume that S occurs exactly k times in each clause. Indeed, if S appears less than k times in a clause, then we can repeat one of its occurrences in that clause. Clauses with no occurrences of S depend only on the structure $\mathbf{A}(I)$ and are true independent of S and hence can be neglected (if such disjuncts are falsified by $\mathbf{A}(I)$, then we do not have a feasible solution).

Given a structure $\mathbf{A}(I)$ with $|\mathbf{A}(I)| = n$ encoding an instance I of a problem in MIN $\mathrm{F}\Pi_1(k)$, we construct an instance $G = (V, E)$ of the MIN k-HYPERVERTEX COVER problem as follows. The set V of vertices of G is the set of all m tuples from the universe of $\mathbf{A}(I)$. Moreover, if $S(\overline{\mathbf{y}}_{i_1}), S(\overline{\mathbf{y}}_{i_2}), \cdots, S(\overline{\mathbf{y}}_{i_k})$ appear in the same clause in the CNF formula, then $\{\overline{\mathbf{y}}_{i_1}, \overline{\mathbf{y}}_{i_2}, \cdots, \overline{\mathbf{y}}_{i_k}\}$ is an edge in G.

Now observe that $S = \{\overline{\mathbf{y}}_{j_1}, \overline{\mathbf{y}}_{j_2}, \cdots, \overline{\mathbf{y}}_{j_t}\}$ is a hypervertex cover for G if and only if by setting $S(\overline{\mathbf{y}}_{j_1}), S(\overline{\mathbf{y}}_{j_2}), \cdots, S(\overline{\mathbf{y}}_{j_t})$ to true we have $(\mathbf{A}(I), S) \models (\forall \mathbf{y})\psi(\mathbf{y}, S)$.

It follows that $\mathcal{Q}$ is A-reducible to MIN k-HYPERVERTEX COVER and so MIN k-HYPERVERTEX COVER is complete for MIN $\mathrm{F}\Pi_1(k)$. □

The approximation properties of the class MIN $\mathrm{F}\Pi_1$ should be contrasted with those of the class RMAX introduced in [PR90]. This is a syntactic subclass of MAX Π_1 that is in some sense the "dual" of MIN $\mathrm{F}\Pi_1$. More formally, RMAX is the class of NP maximization problems with optimum definable as

$$opt_{\mathcal{Q}}(\mathbf{A}) = \max_S \{|S| : \mathbf{A} \models (\forall \mathbf{y})\psi(\mathbf{y}, S)\}$$

where S is a single predicate and ψ is a quantifier-free CNF formula in which all occurrences of S are negative. MAX CLIQUE is the canonical example of a problem in RMAX. Moreover, every problem $\mathcal{Q}$ in this class is self-improvable, i.e., if $\mathcal{Q}$ is approximable, then it has an ϵ-approximation scheme (cf. [PR90]).

Remark 4: We now consider briefly the effect of taking the *A-closure* of the classes MAX Π_n and MAX Σ_n, i.e., all optimization problems that have an A-reduction to a problem in one of these classes. We have seen before that fine distinctions between NP-maximization problems can be made by focusing on their logical definability. It turns out, however, that some of the distinctions manifested in Theorem 2 disappear by passing to A-closures. Indeed, it can be shown that MAX Π_1 contains problems that are complete for the class MAX Π_2 via A-reductions. Such an example is provided by the MAX Number of Satisfiable Formulae (MAX NSF) problem of [PR90]. It should be pointed out that a similar situation holds with NP decision problems. For example, 3-COLORABILITY is expressible using a *strict* Σ^1_1 formula, i.e., an existential second-order formula whose first-order part has universal quantifiers only. It is known that NP problems definable by such formulae have certain special properties that are not denote the union of these classes.

Notice that the second equation in the above definition shows that the class MIN FΠ_1 is a subclass of MIN Σ_1. Notice also that the MIN VERTEX COVER problem is the canonical example of a problem in MIN F$\Pi_1(2)$, since its optimum is given by

$$opt(G) = \min_S\{|S| : G \models (\forall y_1)(\forall y_2)(\neg E(y_1, y_2) \vee S(y_1) \vee S(y_2))\}.$$

By generalizing the vertex cover problem to k-hypergraphs, $k \geq 2$, we can obtain the problem MIN k-HYPERVERTEX COVER. This is a typical example of a problem in MIN F$\Pi_1(k)$.

Definition 5.5: A *k-hypergraph* is a structure $H = (V, E)$ with $E \subseteq V^k$. A *hypervertex cover* is a set $S \subseteq V$ such that for every k-tuple $(v_1, \ldots, v_k)$ in E we have that S contains some v_i.

Notice that a 2-hypergraph can be viewed as an ordinary graph. Moreover, a hypervertex cover for a 2-hypergraph is a vertex cover in the usual sense of the term.

- The MIN k-HYPERVERTEX COVER problem is to find the cardinality of the smallest hypervertex cover in a k-hypergraph. Its optimal is expressed as:

$$opt(G) = \min_S\{|S| : G \models (\forall y_1)\cdots(\forall y_k)(E(y_1, \cdots, y_k) \rightarrow S(y_1) \vee \cdots \vee S(y_k)).$$

The MIN VERTEX COVER problem has a rather straightforward polynomial time 1-approximation algorithm [GJ79] that is based on the idea of maximal matching. By generalizing the notion of maximal matching to hypergraphs, we can obtain a polynomial time 1-approximation algorithm for the MIN k-HYPERVERTEX COVER problem.

Theorem 7: MIN k-HYPERVERTEX COVER is complete for Min F$\Pi_1(k)$, $k \geq 2$, under A-reductions. As a result, every problem in MIN FΠ_1 is approximable.

Proof: Let $\mathcal{Q}$ be a problem in MIN F$\Pi_1(k)$, let I be an instance of it, and let $\mathbf{A}(I)$ be a structure encoding I. Then there is a quantifier-free formula ψ in CNF satisfying the conditions in definition 5.4 such that

$$opt_{\mathcal{Q}}(\mathbf{A}(I)) = \min_S\{|S| : \mathbf{A}(I) \models (\forall \mathbf{y})\psi(\mathbf{y}, S)\}.$$

Let $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_{p(n)}\}$ be the set of possible values for $\mathbf{y}$, where p is a polynomial and $|\mathbf{A}(I)| = n$. Assume also that the arity of S is m. If we let ψ_i be the formula $\psi(\mathbf{y}_i, S)$, then

$$opt_{\mathcal{Q}}(\mathbf{A}(I)) = \min_S\{|S| : \mathbf{A}(I) \models \bigwedge_i \psi_i\}.$$

$S_i(\mathbf{w}'_i)$, where S_i is a predicate symbol from the sequence of symbols $\mathbf{S}$ and $\mathbf{w}'_i$ is an appropriate projection of $\mathbf{w}_i$.

Given an instance I of $\mathcal{Q}$, we construct an instance $t_1(I)$ of MIN 3NT. Corresponding to the output of every gate g in the circuit B_i, we have a variable g in $t_1(I)$. The other variables of $t_1(I)$ are the input variables of the circuit. The disjuncts of $t_1(I)$ are as follows. If g is the output of a NOT gate with input x, then we have $(g \wedge x)$ and $(\overline{g} \wedge \overline{x})$ as disjuncts. If g is the output of an AND gate with inputs x_1, x_2, then we have $(\overline{x}_1 \wedge \overline{x}_2 \wedge g)$ and $(x_1 \wedge x_2 \wedge \overline{g})$ as disjuncts. If g is the output of an OR gate with inputs x_1, x_2, then we have $(x_1 \wedge x_2 \wedge \overline{g})$, $(\overline{x}_1 \wedge x_2 \wedge g)$, $(\overline{x}_1 \wedge \overline{x}_2 \wedge g)$, and $(x_1 \wedge \overline{x}_2 \wedge g)$ as disjuncts. Finally, if g is the output of the circuit B_i, then we have a disjunct (g).

Given any input to the circuit B_i, we can set the boolean values of the intermediate gates such that every disjunct is falsified. The disjuncts are designed such that if g is the output of the AND gate with inputs x_1 and x_2, then setting g to $x_1 \wedge x_2$ will result in falsifying all the disjuncts corresponding to this gate. Similarly, for disjuncts corresponding to OR and NOT gates, if we set the output to the disjunction of the inputs or the negation of the input respectively, then all the disjuncts that correspond to the gate are falsified. Thus, if a truth assignment falsifies $\psi(\mathbf{w}_i, \mathbf{S})$, then we can falsify all the disjuncts corresponding to the circuit B_i. Moreover, if it satisfies $\psi(\mathbf{w}_i, \mathbf{S})$, then the minimum number of disjuncts (corresponding to B_i) satisfied is 1. Hence, $opt_{\mathcal{Q}}(I)$ is equal to the minimum number of satisfiable disjuncts in the instance $t_1(I)$ of 3NT.

In addition, it is straightforward to define the mapping t_2 such that, given an ϵ-approximate truth assignment to the instance $t_1(I)$, we obtain an ϵ-approximate solution to $\mathcal{Q}$. Thus, $\mathcal{Q} \leq_{\mathrm{A}}$ MIN 3NT. □

The preceding Theorem 5 reveals that the pattern of the quantifier prefix does not impact on the approximability of minimization problems, unlike the case of maximization problems. As a result, we have to seek other syntactic features that may imply good approximation properties. We introduce below classes of minimization problems defined by imposing restrictions on the quantifier-free part of formulae and we show that there are natural complete problems for these classes.

Definition 5.4: Let MIN F$\Pi_1(k)$, $k \geq 2$, (F stands for *feasible*) be the class of all minimization problems $\mathcal{Q}$ whose optimum can be expressed as:

$$\begin{aligned} opt_{\mathcal{Q}}(\mathbf{A}) &= \min_S \{|S| : \mathbf{A} \models (\forall \mathbf{y})\psi(\mathbf{y}, S)\} \\ &= \min_S |\{\mathbf{w} : \mathbf{A} \models ((\forall \mathbf{y})\psi(\mathbf{y}, S)) \rightarrow S(\mathbf{w})\}|, \end{aligned}$$

where S is a single predicate, ψ is a quantifier-free CNF formula in which all occurrences of S are positive, and S occurs at most k times in each clause. We also let

$$\text{MIN F}\Pi_1 = \bigcup_k \text{MIN F}\Pi_1(k)$$

Definition 5.2: [CP89] Let $\mathcal{Q}$ and $\mathcal{R}$ be two NP optimization problems. An *approximabilty preserving reduction* (or, A-*reduction*) from $\mathcal{Q}$ to $\mathcal{R}$ is a triple $\tau = (t_1, t_2, c)$ for which the following hold:

- t_1 and t_2 are polynomially computable functions with $t_1 : \mathcal{I}_\mathcal{Q} \to \mathcal{I}_\mathcal{R}$ and $t_2 : \mathcal{I}_\mathcal{R} \times \mathcal{F}_\mathcal{R} \to \mathcal{F}_\mathcal{Q}$.
- c is a function from non-negative rationals to non-negative rationals such that if T is an ϵ-approximate solution for an instance $t_1(I)$ of $\mathcal{R}$, then $t_2(I, T)$ is a $c(\epsilon)$-approximate solution for $\mathcal{Q}$.

If there is an A-reduction form $\mathcal{Q}$ to $\mathcal{R}$, then we say that $\mathcal{Q}$ is A-*reducible* to $\mathcal{R}$ and we write $\mathcal{Q} \leq_{\mathrm{A}} \mathcal{R}$,

The A-reduction defined above is a more relaxed reducibility than the L-reduction defined by Papadimitriou and Yannakakis [PY88]. In the latter the optimum solutions of the two problems $\mathcal{Q}$ and $\mathcal{R}$ are required to be within a constant factor of each other. Although this is the case with many optimization problems, a reduction may preserve approximability (within a constant factor of the optimal) without having this property.

The following propositions follow easily from the definitions.

Proposition 1: if $\mathcal{R}$ is approximable and $\mathcal{Q} \leq_{\mathrm{A}} \mathcal{R}$, then $\mathcal{Q}$ is approximable.

Proposition 2: A-reductions compose.

Definition 5.3: An NP optimization problem $\mathcal{Q}$ is *approximation complete* for a class of problems $\mathcal{C}$ if $\mathcal{Q} \in \mathcal{C}$ and every problem $\mathcal{R} \in \mathcal{C}$ can be A-reduced to $\mathcal{Q}$.

With the necessary definitions behind us, we can now state and prove the following result.

Theorem 6: MIN 3NON-TAUTOLOGY is complete for MIN Σ_0.

Proof: We have shown before that MIN 3NT is in MIN Σ_0. We now prove that every problem in MIN Σ_0 is A-reducible to it. Let $\mathcal{Q}$ be a problem in MIN Σ_0, let I be an instance of it, and let $\mathbf{A}(I)$ be a structure encoding I. Then there is a quantifier-free formula ψ such that

$$opt_\mathcal{Q}(\mathbf{A}(I)) = \min_{\mathbf{S}} |\{\mathbf{w} : \mathbf{A}(I) \models \psi(\mathbf{w}, \mathbf{S})\}|.$$

Let $\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_{p(n)}\}$ be the domain of $\mathbf{w}$, where p is a polynomial and $|\mathbf{A}(I)| = n$. For every $\mathbf{w}_i$ we consider the boolean circuit B_i, composed of gates AND, OR and NOT, that represents the formula $\psi(\mathbf{w}_i, \mathbf{S})$. The inputs to the circuit are of the form

Definition 5.1: [PS82] Let $\mathcal{Q} = (\mathcal{I}_{\mathcal{Q}}, \mathcal{F}_{\mathcal{Q}}, f_{\mathcal{Q}}, opt)$ be an NP optimization problem and let A be an algorithm which, given an instance $I \in \mathcal{I}_{\mathcal{Q}}$, returns a feasible solution $T \in \mathcal{F}_{\mathcal{Q}}$. We say that A is an *ϵ-approximation algorithm* for $\mathcal{Q}$ for some $\epsilon \geq 0$ if

$$\frac{|f_{\mathcal{Q}}(I,T) - opt(I)|}{opt(I)} \leq \epsilon$$

for all instances I. The feasible solution T is said to be an *ϵ-approximate solution* for the instance I. An NP optimization problem is *approximable* if there is a polynomial time ϵ-approximation algorithm for it.

MAX 3SAT, MAX SAT, MIN VERTEX COVER, and TRAVELING SALESMAN with Δ-inequality are important examples of approximable optimization problems. Papadimitriou and Yannakakis [PY88] proved that every problem in MAX Σ_1 is approximable. In contrast to this, we show below that MIN Σ_0 and, a fortiori, MIN Σ_1 contain natural problems that are non-approximable, unless P$\neq$NP. In fact, it turns out that an already familiar problem from the previous section has this property.

Theorem 5: MIN 3NON-TAUTOLOGY is not approximable, unless P = NP .

Proof: Assume that there is an ϵ-approximation algorithm A for MIN 3NT. We show below that A can be used to solve in polynomial time the NON-TAUTOLOGY problem of 3DNF formulae, a problem that is known to be NP complete.

Given an instance ϕ of NON TAUTOLOGY of 3DNF formulae, we create in polynomial time an instance Φ of MIN 3NT as follows: Let x be a variable not occurring in ϕ and let $\overline{x}$ be its negated literal. The formula Φ is a disjunction of $x \vee \overline{x}$ and of n copies of every disjunct of ϕ, where $n > (1+\epsilon)$.

If ϕ is a non-tautology, then $opt_{\mathrm{3NT}}(\Phi) = 1$, because every truth assignment satisfies exactly one of the disjuncts x and $\overline{x}$, and there is a truth assignment under which no disjuncts in any copy of ϕ are satisfied. If ϕ is a tautology, then there is no truth assignment that falsifies every disjunct in ϕ. Hence, in Φ at least one disjunct from each copy of ϕ is satisfied under every truth assignment. Therefore, $opt_{\mathrm{3NT}}(\Phi) \geq n+1$.

It follows that the formula ϕ is a non-tautology if and only if the algorithm A on input Φ returns a value less than or equal to $(1+\epsilon)$. Thus, we have exhibited a polynomial time algorithm for solving an NP-complete problem, which implies that P=NP. $\square$

We now consider an approximation preserving reduction and in Theorem 6 we prove that MIN 3NT is a complete problem for the class MIN Σ_0 under this reduction.

Papadimitriou and Yannakakis [PY88] introduced a notion of L-reduction between optimization problems. Panconesi and Ranjan [PR90] generalized this to the notion of P-reduction. We use here a variant of these reductions introduced by Crescenzi and Panconesi [CP89].

to H_1. Thus, we have $|A_l| \geq k$. Since $\mathbf{y}, \mathbf{z}$ involve u_p and $u_p \notin H_p$, it must be the case that the tuples $\mathbf{y}, \mathbf{z}$ are not elements of A_l. Therefore, we have that $|A_l - A_2| \geq 2$. One of the two elements in $A_l - A_2$ could be $(v_p, v_p, \cdots, v_p)$, but the other element, call it $\mathbf{e}$, will contain both v_i's and u_i's. As a result, $\mathbf{e} \notin A_1$ and, thus, we have demonstrated an element $\mathbf{e}$ that is in $A_l \subset A$, but not in $A_1 \cup A_2$. So, $|A| \geq 2k+1$.

Case b: $t \geq k$. Recall that we have chosen H_1 so that

$$opt(H_1) \neq \min_{\mathbf{S}} |\{(\underbrace{w, \cdots, w}_{m}) : H_1 \models \psi(\underbrace{w, \cdots, w}_{m}, \mathbf{S})\}|.$$

Since $|A_1| = k = opt(H_1)$, it must be the case that

$$A_1 \neq \{(\underbrace{w, \cdots, w}_{m}) : H_1 \models \psi(\underbrace{w, \cdots, w}_{m}, \mathbf{S}^*)\}.$$

Consequently, there exists an element u_q of V_2 that appears in a tuple $\mathbf{y}$ in A_2 and has the property that $(v_q, v_q, \cdots, v_q)$ is not an element of A_1. Let H_s be the subgraph of G induced by $\{u_1, u_2, \cdots, u_{q-1}, v_q, u_{q+1}, \cdots, u_n\}$. Note that H_s is isomorphic to H_1, hence $opt(H_s) = k$ and $|A_s| \geq k$. Note that $\mathbf{y}$ involves u_q, but $u_q \notin H_s$. It follows that $\mathbf{y} \notin A_s$ and, consequently, $|A_s - A_2| \geq 1$. Therefore, $A_s - A_2$ contains $(v_q, v_q, \cdots, v_q)$ or a tuple that has u_i's as some of its components. In either case $A_s - A_2$ contains a tuple that is not an element of $A_1 \cup A_2$. Since $A_s \subset A$, we conclude that $|A| \geq 2k+1$.

Since $\mathbf{S}^*$ was an arbitrarily chosen value of $\mathbf{S}$, we have that

$$opt(G) = \min_{\mathbf{S}} |\{\mathbf{w} : G \models \psi(\mathbf{w}, \mathbf{S})\}| > 2k,$$

which is a contradiction. This completes the proof that MIN VERTEX COVER is not in the class MIN Σ_0. □

Remark 3: An examination of the proofs of Theorems 2 and 4 shows that, when a MAX (MIN) Σ_1 problem is expressed as a MAX (MIN) Π_1 problem, the arity of the sequence of free variables in the resulting Π_1 formula is bigger than the arity of the free variables in the original Σ_1 formula. It should be pointed out that this increase in arity is inevitable in general. Indeed, otherwise one could express the MIN VERTEX COVER problem in MIN Π_1 using a Π_1 formula with a single free variable. In such a case, using arguments similar to those put forth in Part C of Theorem 4, one could also show that MAX CLIQUE is in the class MAX Σ_1, which is false.

5 Approximation Properties of NP Minimization Problems

In this section, we focus on the approximation properties of the minimization classes and contrast them with those of the maximization classes.

We now work with this graph H_1. Let H_1 be (V_1, E_1) with $V_1 = \{v_1, v_2, \cdots, v_n\}$ and assume that the minimum vertex cover of H_1 is of size k. We construct a graph $H_2 = (V_2, E_2)$ with $V_2 = \{u_1, u_2, \cdots, u_n\}$, such that H_2 is isomorphic to H_1 and the isomorphism maps v_i to $u_i, 1 \leq i \leq n$. Using H_1 and H_2, we construct a graph $G = (V, E)$ as follows:

$$\begin{aligned} V &= V_1 \cup V_2 = \{v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_n\} \\ E &= E_1 \cup E_2 \cup \{\{v_i, u_j\} : \{v_i, v_j\} \in E_1\} \cup \{\{u_i, v_j\} : \{v_i, v_j\} \in E_1\}. \end{aligned}$$

Note that any set of n vertices of the form $\{w_1, w_2, \cdots, w_n\}$ with $w_i \in \{v_i, u_i\}$ induces a subgraph of G that is isomorphic to H_1, with the isomorphism mapping w_i to v_i. Consequently, there are 2^n such distinct isomorphic graphs and H_1, H_2 are two of them. We denote these 2^n graphs by $H_1, H_2, \cdots, H_{2^n}$. Note that the minimum vertex cover of each H_j has size k, while the minimum vertex cover of G has size $2k$, i.e.,

$$opt(G) = \min_{\mathbf{S}} |\{\mathbf{w} : G \models \psi(\mathbf{x}, \mathbf{S})\}| = 2k.$$

Let $\mathbf{S}^*$ be an arbitrary value for the sequence of predicate symbols $\mathbf{S}$ and put

$$A = \{\mathbf{w} : G \models \psi(\mathbf{w}, \mathbf{S}^*)\}.$$

We will show that $|A| > 2k$, thereby arriving at a contradiction. Let $\mathbf{S}_j^*$ be the restriction of $\mathbf{S}^*$ to the vertex set of H_j and let

$$A_j = \{\mathbf{w} : H_j \models \psi(\mathbf{w}, \mathbf{S}_j^*)\}, \text{ for } 1 \leq j \leq 2^n.$$

Since the minimum vertex cover of A_j is of size k, we have $|A_j| \geq k$ for $1 \leq j \leq 2^n$.

We now prove that $|A| > 2k$. If $\mathbf{b}$ is an m-tuple from H_j such that $H_j \models \psi(\mathbf{b}, \mathbf{S}_j^*)$, then $G \models \psi(\mathbf{b}, \mathbf{S}^*)$, because Σ_0 formulae are preserved under extensions. So we have, $A_j \subseteq A$, for $1 \leq j \leq 2^n$. In particular $A_1 \subseteq A$ and $A_2 \subseteq A$. Moreover, A_1, A_2 are disjoint, as the vertex sets of H_1, H_2 are disjoint. Therefore, $|A| \geq |A_1| + |A_2| \geq 2k$. We assume that $|A_1| = |A_2| = k$, otherwise we have $|A| > 2k$.

Now we construct one more tuple in A that is not in $A_1 \cup A_2$, which will imply that $|A| > 2k$. Let t denote the number of elements of V_2 that appear in tuples of A_2, i.e.,

$$t = |\{u : u \in V_2 \text{ and } u \text{ appears in a tuple of } A_2\}|.$$

We consider two cases for t, namely $t < k$ and $t \geq k$.

Case a: $t < k$. The k tuples in A_2 are constructed from t distinct elements of V_2. Hence by the pigeonhole principle, there are at least two tuples $\mathbf{y}, \mathbf{z}$ in A_2 that have a common element u_p of V_2 as a component. Now consider the subgraph of G induced by the set $\{u_1, u_2, \cdots, u_{p-1}, v_p, u_{p+1}, \cdots, u_n\}$. Let H_l be this graph, which is isomorphic

Moreover, the sets $\{\mathbf{w} : H_1 \models (\exists \mathbf{y})\psi(\mathbf{w}, \mathbf{y}, \mathbf{S}_1^*)\}$ and $\{\mathbf{w} : H_2 \models (\exists \mathbf{y})\psi(\mathbf{w}, \mathbf{y}, \mathbf{S}_2^*)\}$ are disjoint. Therefore,

$$|\{\mathbf{w} : G \models (\exists \mathbf{y})\psi(\mathbf{w}, \mathbf{y}, \mathbf{S}^*)\}| \geq 2k,$$

which is a contradiction. So, MIN CHROMATIC NUMBER is not in MIN Σ_1.

Part C: In the last part of the proof, we show that MIN VERTEX COVER is not in the class MIN Σ_0.

Towards a contradiction, assume that MIN VERTEX COVER is in the class MIN Σ_0, i.e., assume the optimum is given by

$$opt(G) = \min_{\mathbf{S}} |\{\mathbf{w} : G \models \psi(\mathbf{w}, \mathbf{S})\}|,$$

where ψ is quantifier-free and $\mathbf{w} = (w_1, w_2, \cdots, w_m)$. We distinguish two cases and show that in either case we arrive at a contradiction.

Case 1: Assume that for every graph G the size of the minimum vertex cover is given by

$$opt(G) = \min_{\mathbf{S}} |\{(\underbrace{w, \cdots, w}_{m}) : G \models \psi(\underbrace{w, \cdots, w}_{m}, \mathbf{S})\}|.$$

Let $\psi'(w, \mathbf{S})$ be the formula obtained from ψ by replacing each occurrence of every variable of ψ by w. Notice that $\psi'(w, \mathbf{S})$ is a quantifier-free formula with a single free variable w and has the property that the optimum of the MIN VERTEX COVER can be expressed as

$$opt(G) = \min_{\mathbf{S}} |\{w : G \models \psi'(w, \mathbf{S})\}|.$$

We now exploit the relationship between the problems MIN VERTEX COVER, MAX INDEPENDENT SET, and MAX CLIQUE to arrive at a contradiction. Recall that a set V' is a *vertex cover* of a graph $G = (V, E)$ if and only if $V - V'$ is an independent set in G, i.e., $V - V'$ is a clique in the complement $\overline{G}$ of G. Thus, the minimum vertex cover of G is of size k if and only if its maximum independent set is of size $|V| - k$. In view of the above, for every graph G we have that

$$opt_{\text{IND. SET}}(G) = \max_{\mathbf{S}} |\{w : G \models \neg\psi'(w, \mathbf{S})\}|.$$

This implies that the MAX INDEPENDENT SET problem is in the class MAX Σ_0, since the negation of a quantifier-free formula is also a quantifier-free formula. By using the relationship between MAX INDEPENDENT SET on a graph G and MAX CLIQUE on its complement $\overline{G}$, we conclude that MAX CLIQUE is in the class MAX Σ_0. This, however, contradicts one of the results in [PR90] (cf. also Theorem 2).

Case 2: Assume that there is some graph, call it H_1, for which the size of the minimum vertex cover satisfies

$$opt(H_1) \neq \min_{\mathbf{S}} |\{(\underbrace{w, \cdots, w}_{m}) : H_1 \models \psi(\underbrace{w, \cdots, w}_{m}, \mathbf{S})\}|.$$

$\mathbf{w}$, then again $\phi(\mathbf{w}, \mathbf{x}^*, \mathbf{S}, R)$ is satisfied. Thus, $(\mathbf{w}, \mathbf{x}^*) \in V(R)$ for this $\mathbf{x}^*$. Therefore, if $\mathbf{w} \in U$, then for every R there is a tuple $\mathbf{x}^*$ such that $(\mathbf{w}, \mathbf{x}^*) \in V(R)$. So, $|U| \leq |V(R)|$ for all R and, as a result,

$$|U| \leq \min_R |V(R)|.$$

In the second step, for each $\mathbf{w}$ in U let $\mathbf{x}_\mathbf{w}$ be a fixed witness of $\mathbf{w}$ and let

$$R_0 = \{(\mathbf{w}, \mathbf{x}) : \mathbf{w} \in U \text{ and } \mathbf{x} = \mathbf{x}_\mathbf{w}\}.$$

It is now easy to verify that

$$|U| = |R_0| \text{ and } R_0 = V(R_0).$$

It now follows that $|U| = |V(R_0)|$ and, as a result, we have that $|U| = \min_R |V(R)|$. Since $\mathbf{S}$ was an arbitrary sequence of predicates, we conclude that

$$opt_\mathcal{Q}(\mathbf{A}) = \min_\mathbf{S} |U(\mathbf{S})| = \min_{\mathbf{S},R} |V(\mathbf{S}, R)|.$$

This establishes that $\mathcal{Q}$ is in MIN Π_1 and, consequently, the class MIN Σ_2 is contained in the class MIN Π_1.

Part B: In this part of the proof we show that MIN CHROMATIC NUMBER is in MIN Π_1, but not in MIN Σ_1.

We have already seen that MIN CHROMATIC NUMBER is in the class MIN Σ_2 and hence, by what we proved in Part A, it is in the class MIN Π_1. We now show that MIN CHROMATIC NUMBER is not in the class MIN Σ_1. Towards a contradiction, assume that it is in the class MIN Σ_1. Therefore, there is a quantifier-free formula $\psi(\mathbf{w}, \mathbf{y}, \mathbf{S})$ such that for every graph G

$$opt(G) = \min_\mathbf{S} |\{\mathbf{w} : G \models (\exists \mathbf{y})\psi(\mathbf{w}, \mathbf{y}, \mathbf{S})\}|.$$

Let H_1 be a graph with $opt(H_1) = k$ and let H_2 be an isomorphic copy of H_1. We construct a graph G by taking the disjoint union of H_1 and H_2. Note that $opt(G) = k$ and there is an $\mathbf{S}^*$ such that

$$|\{\mathbf{w} : G \models (\exists \mathbf{y})\psi(\mathbf{w}, \mathbf{y}, \mathbf{S}^*)\}| = k.$$

Let $\mathbf{S}_1^*$ and $\mathbf{S}_2^*$ be the restrictions of $\mathbf{S}^*$ to the vertex sets of H_1 and H_2 respectively. If $\mathbf{b}$ is a tuple from H_i, $i = 1, 2$, such that $H_i \models (\exists \mathbf{y})\psi(\mathbf{b}, \mathbf{y}, \mathbf{S}_1^*)$, then it is also the case that $G \models (\exists \mathbf{y})\psi(\mathbf{b}, \mathbf{y}, \mathbf{S}^*)$, because existential formulae are preserved under extensions. But,

$$|\{\mathbf{w} : H_i \models (\exists \mathbf{y})\psi(\mathbf{w}, \mathbf{y}, \mathbf{S}_i^*)\}| \geq k, \quad \text{for } i = 1, 2.$$

It now follows that if G is a graph, then

$$opt(G) = \min_S |\{c : G \models \psi(S) \to (\exists x)S(x,c)\}|.$$

Thus, MIN CHROMATIC NUMBER is in MIN Σ_2.

The next result clarifies the exact relationship between the four classes of minimization problems.

Theorem 4: The class MIN Σ_2 is contained in the class MIN Π_1. As a result,

$$\text{MIN } \Sigma_0 \subset \text{MIN } \Sigma_1 \subset \text{MIN } \Pi_1 = \text{MIN } \Sigma_2 = \text{MIN } \mathcal{PB}.$$

Moreover, these containments are strict. In particular,

- MIN CHROMATIC NUMBER is in MIN Π_1, but not in MIN Σ_1.
- MIN VERTEX COVER is in MIN Σ_1, but not in MIN Σ_0.

Proof: We give this proof in three parts.

Part A: In this part we show that MIN Σ_2 is a subclass of MIN Π_1.

Let $\mathcal{Q}$ be a problem in MIN Σ_2. Then there is a quantifier-free formula $\psi(\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{S})$ such that for every finite structure $\mathbf{A}$ that is an instance of $\mathcal{Q}$ we have

$$opt_{\mathcal{Q}}(\mathbf{A}) = \min_{\mathbf{S}} |\{\mathbf{w} : \mathbf{A} \models (\exists \mathbf{x})(\forall \mathbf{y})\psi(\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{S})\}|,$$

Let

$$U(\mathbf{S}) = \{\mathbf{w} : \mathbf{A} \models (\exists \mathbf{x})(\forall \mathbf{y})\psi(\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{S})\}$$

and

$$V(\mathbf{S}, R) = \{(\mathbf{w}, \mathbf{x}^*) : \mathbf{A} \models \phi(\mathbf{w}, \mathbf{x}^*, \mathbf{S}, R)\},$$

where ϕ is the following Π_1 formula

$$\begin{aligned} \phi(\mathbf{w}, \mathbf{x}^*, \mathbf{S}, R) \equiv\ & [(\neg R(\mathbf{w}, \mathbf{x}^*)) \to (\forall \mathbf{y})\psi(\mathbf{w}, \mathbf{x}^*, \mathbf{y}, \mathbf{S})] \wedge \\ & [((\exists \mathbf{x})R(\mathbf{w}, \mathbf{x})) \to R(\mathbf{w}, \mathbf{x}^*)] \end{aligned}$$

We prove below that $|U(\mathbf{S})| = \min_R |V(\mathbf{S}, R)|$ for all $\mathbf{S}$. In what follows we fix the sequence of predicates $\mathbf{S}$ and, for simplicity, we write U for the set $U(\mathbf{S})$ and $V(R)$ for the set $V(\mathbf{S}, R)$. If $\mathbf{A} \models (\forall \mathbf{y})\psi(\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{S})$, then we say that $\mathbf{x}$ is a *witness of* $\mathbf{w}$ *relative to* $\mathbf{S}$.

We prove that $|U| = \min_R |V(R)|$ in two steps. First, observe that if $\mathbf{w} \in U$, then $\mathbf{w}$ has a witness. If R is such that $\neg(\exists \mathbf{x})R(\mathbf{w}, \mathbf{x})$, then the formula $\phi(\mathbf{w}, \mathbf{x}^*, \mathbf{S}, R)$ is satisfied for every witness $\mathbf{x}^*$ of $\mathbf{w}$. Thus, $(\mathbf{w}, \mathbf{x}^*) \in V(R)$ for every witness $\mathbf{x}^*$ of $\mathbf{w}$. On the other hand, if R is such that $R(\mathbf{w}, \mathbf{x}^*)$ for some $\mathbf{x}^*$, which may not be a witness of

By restricting the quantifier prefix $\exists^*\forall^*$ of Σ_2 formulae, we obtain the classes MIN Π_1, MIN Σ_1 and MIN Σ_0. It is obvious that:

$$\begin{array}{ccc} & \text{MIN } \Sigma_1 & \\ \text{MIN } \Sigma_0 & & \text{MIN } \Sigma_2 = \text{MIN } \mathcal{PB} \\ & \text{MIN } \Pi_1 & \end{array}$$

We give below examples of some natural problems in these classes that will be used in the sequel.

We begin by presenting MIN 3NON-TAUTOLOGY, which is an optimization problem in MIN Σ_0 that arises from the NP-complete problem NON-TAUTOLOGY of 3DNF formulae [GJ79]: Given a boolean formula in disjunctive normal form with three literals per disjunct (3DNF), is there a truth assignment that makes this formula false?

• MIN 3NON-TAUTOLOGY (3NT): Given a boolean formula in 3DNF, find the minimum number of satisfiable disjuncts.

We view every instance I of MIN 3NT as a finite structure $\mathbf{A}(I)$ with four ternary predicates D_0, D_1, D_2, D_3, where $D_i(w_1, w_2, w_3)$ is true if and only if the set $\{w_1, w_2, w_3\}$ is a disjunct with $w_1, \cdots, w_i$ appearing as negative literals and $w_{i+1}, \cdots, w_3$ appearing as positive literals, $0 \leq i \leq 3$. The optimum of 3NT is given by

$$opt(I) = \min_S |\{(w_1, w_2, w_3) : \mathbf{A} \models \phi(w_1, w_2, w_3, S)\}|,$$

where $\phi(w_1, w_2, w_3)$ is the following quantifier-free formula:

$$(D_0(w_1, w_2, w_3) \wedge S(w_1) \wedge S(w_2) \wedge S(w_3)) \vee (D_1(w_1, w_2, w_3) \wedge \neg S(w_1) \wedge S(w_2) \wedge S(w_3)) \vee$$

$$(D_2(w_1, w_2, w_3) \wedge \neg S(w_1) \wedge \neg S(w_2) \wedge S(w_3)) \vee (D_3(w_1, w_2, w_3) \wedge \neg S(w_1) \wedge \neg S(w_2) \wedge \neg S(w_3)).$$

• MIN VERTEX COVER problem is a natural problem in the class MIN Σ_1. On any graph G the optimum is given by

$$\begin{aligned} opt(G) &= \min_S\{|S| : G \models (\forall y_1)(\forall y_2)\,[\,E(y_1, y_2) \rightarrow (S(y_1) \vee S(y_2))\,]\,\} \\ &= \min_S |\{x : G \models \;[\,(\forall y_1)(\forall y_2)[E(y_1, y_2) \rightarrow (S(y_1) \vee S(y_2))]\,] \;\rightarrow S(x)\}|\,. \end{aligned}$$

• MIN CHROMATIC NUMBER is an important polynomially bounded minimization problem (cf. [GJ79]). Theorem 3 implies that MIN CHROMATIC NUMBER in the class MIN Σ_2. We exhibit below a Σ_2 formula that establishes this fact directly.

Consider first the following Π_2 sentence $\psi(S)$ asserting that S is a coloring:

$$\begin{aligned} \psi(S) \;\equiv\; & (\forall x)(\exists c)S(x, c) \;\wedge\; (\forall x)(\forall c_1)(\forall c_2)[S(x, c_1) \wedge S(x, c_2) \rightarrow (c_1 = c_2)] \\ & \wedge (\forall x)(\forall y)(\forall c_1)(\forall c_2)[E(x, y) \wedge S(x, c_1) \wedge S(y, c_2) \rightarrow (c_1 \neq c_2)]. \end{aligned}$$

4 Polynomially Bounded NP Minimization Problems

The logical definability of NP minimization problems has not been explored in the literature so far. We undertake this investigation here and unveil a strikingly different picture from the one for NP maximization problems. The next result should be contrasted with Theorem 1 in Section 3.

Theorem 3: Let σ be a vocabulary and let $\mathcal{Q}$ be an NP minimization problem with finite structures $\mathbf{A}$ over σ as instances. Then $\mathcal{Q}$ is a polynomially bounded NP minimization problem if and only if there is a first order formula $\phi(\mathbf{w}, \mathbf{S})$ with predicate symbols among those in σ and $\mathbf{S}$ such that for every instance $\mathbf{A}$ of $\mathcal{Q}$

$$opt_{\mathcal{Q}}(\mathbf{A}) = \min_{\mathbf{S}} |\{\mathbf{w} : \mathbf{A} \models \phi(\mathbf{w}, \mathbf{S})\}|.$$

Moreover, $\phi(\mathbf{w}, \mathbf{S})$ can always be taken to be a Σ_2 formula and, consequently,

$$\text{MIN } \mathcal{PB} = \text{MIN } \Sigma_2 = \text{MIN } \Sigma_n, \quad n > 2.$$

Proof: Following the same arguments as in Theorem 1, we can show that if $\mathcal{Q}$ is a polynomially bounded NP minimization problem, then there is a Π_2 formula $\psi(S^*, W)$ such that

$$opt_{\mathcal{Q}}(\mathbf{A}) = \min_{\mathbf{S}^*, W}\{|W| : \mathbf{A} \models \psi(\mathbf{S}^*, W)\}$$

It follows that

$$opt_{\mathcal{Q}}(\mathbf{A}) = \min_{\mathbf{S}^*, W} |\{\mathbf{w} : \mathbf{A} \models \psi(\mathbf{S}^*, W) \rightarrow W(\mathbf{w})\}|$$

Let $\mathbf{S}$ denote the sequence $(\mathbf{S}^*, W)$ and let $\phi(\mathbf{w}, \mathbf{S})$ be the Σ_2 formula

$$\psi(\mathbf{S}^*, W) \rightarrow W(\mathbf{w}).$$

We can now conclude that

$$opt_{\mathcal{Q}}(\mathbf{A}) = \min_{\mathbf{S}} |\{\mathbf{w} : \mathbf{A} \models \phi(\mathbf{w}, \mathbf{S})\}|.$$

□

Remark 2: Notice that, unlike the case of maximization problems, if

$$opt_{\mathcal{Q}}(\mathbf{A}) = \min_{\mathbf{S}^*, W}\{|W| : \mathbf{A} \models \psi(\mathbf{S}^*, W)\},$$

then it is not true that

$$opt_{\mathcal{Q}}(\mathbf{A}) = \min_{\mathbf{S}^*, W} |\{\mathbf{w} : \mathbf{A} \models W(\mathbf{w}) \wedge \psi(\mathbf{S}^*, W)\}|,$$

because the minimum cardinality of the above set is zero, which occurs when W is empty. This explains the "dual" behavior in logical definability between maximization and minimization problems, viz. MAX $\mathcal{PB}$ = MAX Π_2, while MIN $\mathcal{PB}$ = MIN Σ_2.

We now define a structure $\mathbf{A} = (X, C, P, N)$ as follows.

$$\begin{aligned} X &= \bigcup_i^n X_i, \quad C = \bigcup_i^n C_i, \\ P &= \{(x_u^i, x_v^j) : P_1(x_u^1, x_v^1), 1 \le u, v, i, j \le n\}, \\ N &= \{(x_u^i, x_v^j) : N_1(x_u^1, x_v^1), 1 \le u, v, i, j \le n\}. \end{aligned}$$

It can be seen that $\mathbf{A}$ encodes an instance of MAX SAT. Also, observe that $|C| = n|C_1| \le n(n-1)$, as the structure $\mathbf{A}_1$ has at least one variable. Therefore, $opt(\mathbf{A}) \le n(n-1)$. We will arrive at a contradiction by showing that $opt(\mathbf{A}) \ge n^2$.

For $1 \le l \le t$, let

$$\begin{aligned} \mathcal{S}_l^* &= \{(x_{u_1}^{i_1}, x_{u_2}^{i_2}, \cdots, x_{u_{\alpha[l]}}^{i_{\alpha[l]}}) : \ \mathcal{S}_l^*(x_{u_1}^1, x_{u_2}^1, \cdots, x_{u_{\alpha[l]}}^1), \quad \text{where} \\ &\qquad 1 \le i_1, \cdots, i_{\alpha[l]} \le n \ \text{ and } \ 1 \le u_1, \cdots, u_{\alpha[l]} \le n\}, \end{aligned}$$

and let $\mathbf{S}^*$ denote the sequence $(\mathcal{S}_1^*, \mathcal{S}_2^*, \cdots, \mathcal{S}_t^*)$. We will show that $|V| \ge n^2$, where

$$V = \{(w_1, \cdots, w_m) : \mathbf{A} \models \psi(w_1, \cdots, w_m, \mathcal{S}^*)\}.$$

From the hypothesis of Case 2, we know that there is a tuple $\mathbf{e}$ in V_1 with at least two distinct components x_p^1 and x_q^1. For every i, j with $1 \le i, j \le n$, let $\mathbf{e}_{i,j}$ be obtained from $\mathbf{e}$ by replacing every occurrence of x_p^1 by x_p^i and every occurrence of x_q^1 by x_q^j. Also let $\mathbf{A}_{i,j}$ denote the substructure of $\mathbf{A}$ with universe

$$\{x_1^1, \cdots, x_{p-1}^1, x_p^i, x_{p+1}^1, \cdots, x_{q-1}^1, x_q^j, x_{q+1}^1, \cdots, x_n\}.$$

It is clear that $\mathbf{A}_{i,j}$ is isomorphic to $\mathbf{A}_1$. Moreover, the restriction of $\mathcal{S}^*$ to the above set is a sequence of predicates isomorphic to $\mathbf{S}^*$, where the isomorphism maps x_p^i to x_p^1, maps x_q^i to x_q^1 and is the identity on the rest of the elements. Let $\mathcal{S}_{i,j}^*$ denote the restriction of $\mathcal{S}^*$ to universe of $\mathbf{A}_{i,j}$.

Observe that $\mathbf{A}_{i,j} \models \psi(\mathbf{e}_{i,j}, \mathcal{S}_{i,j}^*)$ for $1 \le i, j \le n$. Since Σ_0 sentences are preserved under extensions, it is also true that $\mathbf{A} \models \psi(\mathbf{e}_{i,j}, \mathcal{S}^*)$ for $1 \le i, j \le n$. As there are n^2 distinct such elements $\mathbf{e}_{i,j}$, we have that $|V| \ge n^2$. It follows that $opt(\mathbf{A}) \ge n^2$, which is a contradiction. The proof that MAX SAT is not in the class MAX Σ_0 is now complete. □

Remark 1: The class MAX Σ_2 is another collection of maximization problems that contains both MAX Σ_1 and MAX Π_1. The proof of Theorem 2 also yields that MAX Σ_2 = MAX Π_1.

C, P, N and $S_1, \cdots, S_t$ in $\mathbf{S}$ are amongst the following:

$$C(w), \neg C(w), P(w,w), \neg P(w,w), N(w,w), \neg N(w,w),$$
$$S_l(\underbrace{w,\cdots,w}_{\alpha[l]}), \neg S_l(\underbrace{w,\cdots,w}_{\alpha[l]}), 1 \leq l \leq t,$$

where $\alpha[l]$ is the arity of S_l. For every instance I encoded by a finite structure $\mathbf{A}(I) = (X, C, P, N)$, it is the case that $\mathbf{A}(I) \not\models P(x,x)$ and $\mathbf{A}(I) \not\models N(x,x)$, for all $x \in X$, because the first arguments of P, N refer to a clause, the second to a variable and the variables are different from the clauses. Let ψ'' be the formula obtained from $\psi'(w, \mathbf{S})$ by replacing each occurrence of $P(w,w)$, $N(w,w)$ by the logical constant $FALSE$, and each occurrence of $\neg P(w,w)$, $\neg N(w,w)$ by the logical constant $TRUE$. Then we have that for every instance I

$$opt(\mathbf{A}(I)) = \max_{\mathbf{S}} |\{w : \mathbf{A}(I) \models \psi''(w, \mathbf{S})\}.$$

Let I_1, I_2 be two instances of MAX SAT, each having the same number of variables and the same number of clauses, but differing in the maximum number of satisfiable clauses. Without loss of generality, we can find structures $\mathbf{A}(I_1) = (X_1, C_1, P_1, N_1)$ and $\mathbf{A}(I_2) = (X_2, C_2, P_2, N_2)$ encoding I_1, I_2 respectively, such that $X_1 = X_2$ and $C_1 = C_2$. Since $\psi''(w, \mathbf{S})$ does not have any occurrences of the symbols P, N, we have

$$\{w : \mathbf{A}(I_1) \models \psi''(w, \mathbf{S})\} = \{w : \mathbf{A}(I_2) \models \psi''(w, \mathbf{S})\}.$$

for all values of $\mathbf{S}$. Therefore,

$$opt(\mathbf{A}(I_1)) = opt(\mathbf{A}(I_2)),$$

which is a contradiction.

Case 2: Assume that there is some instance I_1, such that its encoding by the structure $\mathbf{A}(I_1) = (X_1, C_1, P_1, N_1)$ satisfies

$$opt(\mathbf{A}(I_1)) \neq \max_{\mathbf{S}} |\{(\underbrace{w,\cdots,w}_{m}) : \mathbf{A}(I_1) \models \psi(\underbrace{w,\cdots,w}_{m}, \mathbf{S})\}|.$$

For simplicity, we write $\mathbf{A}_1$ for the structure $\mathbf{A}(I_1)$.

Let $\mathbf{S}^*$ be a sequence of predicates $(S_1^*, S_2^*, \cdots, S_t^*)$ that realizes $opt(\mathbf{A}_1)$, i.e.,

$$opt(\mathbf{A}_1) = |\{(w_1, \cdots, w_m) : \mathbf{A}_1 \models \psi(w_1, \cdots, w_m, \mathbf{S}^*)\}|.$$

Let

$$V_1 = \{(w_1, \cdots, w_m) : \mathbf{A}_1 \models \psi(w_1, \cdots, w_m, \mathbf{S}^*)\}$$

and assume that $X_1 = \{x_1^1, x_2^1, \cdots, x_n^1\}$. We now construct $n-1$ additional structures, $\mathbf{A}_2, \cdots, \mathbf{A}_n$, where $\mathbf{A}_i = (X_i, C_i, P_i, N_i)$ with $X_i = \{x_1^i, x_2^i, \cdots, x_n^i\}, 2 \leq i \leq n$, such that they are all isomorphic to $\mathbf{A}_1$ via the mapping x_u^i to x_u^1, for $1 \leq i, u \leq n$.

from G by deleting a_i and all edges incident to it. Assume that the maximum value in the above expression occurs at $\mathbf{S} = \mathbf{S}^*$. Let $\mathbf{S}_i^*$ be the restriction of $\mathbf{S}^*$ to the vertex set $\{a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n\}$ of H_i. Since $opt(H_i) = n - i$, we have that

$$|\{\mathbf{w} : H_i \models (\forall \mathbf{y})\psi(\mathbf{w}, \mathbf{y}, \mathbf{S}_i^*)\}| \leq n - i.$$

Since universal formulae are preserved under substructures, we have that if $\mathbf{b}$ is an m-tuple from H_i such that $G \models (\forall \mathbf{y})\psi(\mathbf{b}, \mathbf{y}, \mathbf{S}^*)$, then $H_i \models (\forall \mathbf{y})\psi(\mathbf{b}, \mathbf{y}, \mathbf{S}_i^*)$. Therefore, each a_i occurs in at least i tuples in the set $\{\mathbf{w} : G \models (\forall \mathbf{y})\psi(\mathbf{w}, \mathbf{y}, \mathbf{S}^*)\}$. As a result, the total number of occurrences of all a_i's in this set is at least $(\sum_{i=1}^{i=n/2} i) > nm$, since $n > 8m + 1$.

On the other hand, since $\mathbf{w}$ ranges over tuples of arity m and the cardinality of the set $\{\mathbf{w} : G \models (\forall \mathbf{y})\psi(\mathbf{w}, \mathbf{y}, \mathbf{S}^*)\}$ is n, the total number of occurrences of all a_i's in this set is at most nm. Thus, we have arrived at a contradiction.

Part C: Panconesi and Ranjan [PR90] showed that MAX CLIQUE is in the class MAX Π_1, but not in the class MAX Σ_1.

Part D: We have seen before that MAX SAT is in the MAX Σ_1. In this part of the proof we show that MAX SAT is not in the class MAX Σ_0. Let I be an instance of SAT and let $\mathbf{A}(I) = (X, C, P, N)$ be its encoding as a finite structure. Recall that X consists of the variables and the clauses of I, while the predicate C separates the clauses from the variables. Towards a contradiction, assume that MAX SAT is in the class MAX Σ_0. Therefore, there is a quantifier-free formula $\psi(\mathbf{w}, \mathbf{S})$ such that for every finite structure $\mathbf{A}(I)$ encoding an instance I of MAX SAT we have that

$$opt(\mathbf{A}(I)) = \max_{\mathbf{S}} |\{\mathbf{w} : \mathbf{A}(I) \models \psi(\mathbf{w}, \mathbf{S})\}|,$$

where $\mathbf{w}$ ranges over m-tuples $(w_1, w_2, \cdots, w_m)$ and $\mathbf{S} = (S_1, \cdots, S_l)$. We distinguish two cases and show that in either case we arrive at a contradiction.

Case 1: Assume that, for every structure $\mathbf{A}(I)$ encoding an instance I the maximum number of clauses satisfiable is given by

$$opt(\mathbf{A}(I)) = \max_{\mathbf{S}} |\{(\underbrace{w, \cdots, w}_{m}) : \mathbf{A}(I) \models \psi(\underbrace{w, \cdots, w}_{m}, \mathbf{S})\}|.$$

Let $\psi'(w, \mathbf{S})$ be the formula obtained from ψ by replacing each occurrence of every free variable by w. It is clear that

$$opt(\mathbf{A}(I)) = \max_{\mathbf{S}} |\{w : \psi'(w, \mathbf{S})\}|.$$

Since ψ is a quantifier-free formula, ψ' is also a quantifier-free formula whose only variable is w. As a result, in $\psi'(w, \mathbf{S})$ the only occurrences of the predicate symbols

- MAX CONNECTED COMPONENT is in MAX Π_2, but not in MAX Π_1.
- MAX CLIQUE is in MAX Π_1, but not in MAX Σ_1 ([PR90]).
- MAX SAT is in MAX Σ_1, but not in MAX Σ_0.

Proof: We give this proof in four parts.

Part A: In this part, we prove that MAX Σ_1 is contained in the class MAX Π_1. Let $\mathcal{Q}$ be a MAX Σ_1 problem and $\mathbf{A}$ be a finite structure that is an instance of $\mathcal{Q}$. Thus,

$$opt_{\mathcal{Q}}(\mathbf{A}) = \max_{\mathbf{S}} |\{\mathbf{w} : \mathbf{A} \models (\exists \mathbf{x})\psi(\mathbf{w}, \mathbf{x}, \mathbf{S})\}|,$$

where ψ is quantifier-free. If $\mathbf{A} \models \psi(\mathbf{w}, \mathbf{x}^*, \mathbf{S})$, then we say that $\mathbf{x}^*$ is a *witness of* $\mathbf{w}$ *relative to* $\mathbf{S}$.

Consider now the sets

$$U(\mathbf{S}) = \{\mathbf{w} : \mathbf{A} \models (\exists \mathbf{x})\psi(\mathbf{w}, \mathbf{x}, \mathbf{S})\}$$

and

$$\begin{aligned} V(\mathbf{S}, R) = \{(\mathbf{w}, \mathbf{x}^*) : \mathbf{A} \models \psi(\mathbf{w}, \mathbf{x}^*, \mathbf{S}) \wedge R(\mathbf{w}, \mathbf{x}^*) \wedge \\ (\forall \mathbf{x}_1)(\forall \mathbf{x}_2)((R(\mathbf{w}, \mathbf{x}_1) \wedge R(\mathbf{w}, \mathbf{x}_2)) \rightarrow \mathbf{x}_1 = \mathbf{x}_2)\} \end{aligned}$$

Intuitively, a pair $(\mathbf{w}, \mathbf{x}^*)$ is in the set $V(\mathbf{S}, R)$ if $\mathbf{x}^*$ is a witness of $\mathbf{w}$ relative to $\mathbf{S}$ and $\mathbf{x}^*$ is the only tuple $\mathbf{x}$ such that the pair $(\mathbf{w}, \mathbf{x})$ is in R. It is now easy to verify that for every fixed sequence of relations $\mathbf{S}$ we have that

$$|U(\mathbf{S})| = \max_{R} |V(\mathbf{S}, R)|$$

and, as a result,

$$opt(\mathbf{A}) = \max_{\mathbf{S}} |U(\mathbf{S})| = \max_{\mathbf{S}, R} |V(\mathbf{S}, R)|.$$

Since $V(\mathbf{S}, R)$ is defined using a Π_1 formula, it follows that $\mathcal{Q} \in$ MAX Π_1 and, consequently, the class MAX Σ_1 is a subset of the class MAX Π_1.

Part B: We showed earlier that MCC is in the class MAX Π_2. In this part of the proof we show that MCC is not in the class MAX Π_1.

Towards a contradiction, assume that the optimum of MCC is given by

$$opt(G) = \max_{\mathbf{S}} |\{\mathbf{w} : G \models (\forall \mathbf{y})\psi(\mathbf{w}, \mathbf{y}, \mathbf{S})\}|,$$

where ψ is quantifier-free and $\mathbf{w}$ ranges over tuples of arity m.

Let G be a graph that is a path with vertices $\{a_1, \cdots, a_n\}$, for some $n > 8m + 1$, and edges $\{a_i, a_{i+1}\}$, $1 \leq i \leq n-1$. Consider the subgraphs H_i, $1 \leq i \leq \lfloor n/2 \rfloor$, obtained

• MAX CLIQUE is in the class MAX Π_1 (cf. [PR90]). Indeed, for MAX CLIQUE we have that

$$opt(G) = \max_S |\{w : G \models S(w) \wedge (\forall y_1)(\forall y_2)[(S(y_1) \wedge S(y_2) \wedge (y_1 \neq y_2)) \rightarrow E(y_1, y_2)]\ \}|.$$

• MAX CONNECTED COMPONENT (MCC): Given an undirected graph G, find the size of the largest connected component in G.

Notice that actually MCC is an optimization problem on graphs that can be solved in polynomial time. This problem will be of particular interest to us in the sequel.

Although Theorem 1 implies that MCC is in the class MAX Π_2, it is not obvious how to establish this directly. In what follows we produce a Π_2 formula ϕ that defines MCC in our framework.

In addition to a binary relation symbol E for the edges of the graph, the formula ϕ will involve the relation symbols $C, E, P, \leq, \mathcal{Z}$. The intuition behind these is as follows: C is a unary relation symbol that represents the vertices of a connected component; $\leq$ is a binary relation that will vary over total orders on the vertices of the graph; P is a ternary relation symbol; $P(x, y, k)$ indicates that the shortest path from x to y is of length k, where the integer k is encoded by the k^{th} element of the total order $\leq$; finally, $\mathcal{Z}$ is a unary predicate representing the smallest element of the total order $\leq$ ($\mathcal{Z}$ for zero).

Let $\phi_1(\leq)$ be a formula asserting that $\leq$ is a total order and let $\phi_2(\mathcal{Z})$ be a formula asserting that $\mathcal{Z}$ is a singleton set containing the smallest element of $\leq$. Let also $pred(x, y)$ be a formula asserting that y is the predecessor of x under the above order. We leave it to the reader to verify that $\phi_1(\leq)$ and $pred(x, y)$ can be expressed as Π_1 formulae, while $\phi_2(\mathcal{Z})$ can be written as a conjunction of Π_1 and Σ_1 formulae. We are now ready to demonstrate that MCC is in the class MAX Π_2. Indeed, its optimum value on a graph G is given as

$$\begin{aligned} opt(G) = \max_{(C,P,\leq,\mathcal{Z})} |\{w : C(w) \wedge \phi_1(\leq) \wedge \phi_2(\mathcal{Z}) \wedge \\ (\forall x)(\forall y)((C(x) \wedge C(y)) \rightarrow (\exists z)P(x, y, z)) \wedge \\ (\forall x)(\forall y)(\forall v)(\forall v')[(P(x, y, v) \wedge \neg\mathcal{Z}(v) \wedge pred(v, v')) \rightarrow \\ ((\exists z)P(x, z, v') \wedge E(z, y))] \wedge \\ (\forall x)(\forall y)(\forall v)((P(x, y, v) \wedge \mathcal{Z}(v)) \rightarrow (x = y))\ \}| \end{aligned}$$

The next result clarifies the relationship between the above classes of maximization problems and shows that the polynomially bounded NP maximization problems form a hierarchy with exactly four distinct levels.

Theorem 2: The class MAX Σ_1 is contained in the class MAX Π_1. As a result,

$$\text{MAX } \Sigma_0 \subset \text{MAX } \Sigma_1 \subset \text{MAX } \Pi_1 \subset \text{MAX } \Pi_2.$$

Moreover, this sequence of containments is strict. In particular, or, equivalently,

$$\mathrm{opt}_{\mathcal{Q}}(\mathbf{A}) = \max_{\mathbf{S}^*, W} |\{\mathbf{w} : \mathbf{A} \models W(\mathbf{w}) \wedge \psi(\mathbf{S}^*, W)\}|.$$

Let $\mathbf{S}$ denote the sequence $(\mathbf{S}^*, W)$ and let $\phi(\mathbf{w}, \mathbf{S})$ be the formula $W(\mathbf{w}) \wedge \psi(\mathbf{S}^*, W)$. It follows that

$$\mathrm{opt}_{\mathcal{Q}}(\mathbf{A}) = \max_{\mathbf{S}} |\{\mathbf{w} : \mathbf{A} \models \phi(\mathbf{w}, \mathbf{S})\}|.$$

Moreover, $\phi(\mathbf{w}, \mathbf{S})$ can be chosen to be a Π_2 formula, because Fagin's characterization of NP [Fag74] holds with a Π_2 formula $\psi(\mathbf{w}, \mathbf{S}^*)$. $\square$

Theorem 1 shows that MAX Π_2 is the entire class MAX $\mathcal{PB}$ of polynomially bounded NP maximization problems. By restricting the quantifier prefix $\forall^*\exists^*$ of Π_2 formulae, we obtain the class MAX Π_1 of [PR90], and the classes MAX Σ_1 = MAX NP and MAX Σ_0 = MAX SNP of [PY88]. It is clear that we have the following containments between these four classes:

	MAX Σ_1	
MAX Σ_0		MAX Π_2 = MAX $\mathcal{PB}$
	MAX Π_1	

We now give examples of natural problems in these classes.

• MAX 3SAT is a problem in the class MAX Σ_0 (cf. [PY88]). This problem asks for the maximum number of clauses that can be satisfied in a given Boolean formula in conjunctive normal form (CNF) with three literals per clause. We view every instance I of MAX 3SAT as a finite structure $\mathbf{A}(I)$ with universe the variables of the formula and with four ternary predicates C_0, C_1, C_2, C_3. Under this encoding, $C_i(w_1, w_2, w_3)$ is true if and only if $\{w_1, w_2, w_3\}$ is a clause with $w_1, \cdots, w_i$ appearing as negative literals and $w_{i+1}, \cdots, w_3$ appearing as positive literals, $0 \leq i \leq 3$. The optimum of 3SAT is given by

$$opt(\mathbf{A}(I)) = \max_{S} |\{(w_1, w_2, w_3) : \mathbf{A} \models \phi(w_1, w_2, w_3, S)\}|,$$

where $\phi(w_1, w_2, w_3)$ is the formula

$$C_0(w_1, w_2, w_3) \wedge (S(w_1) \vee S(w_2) \vee S(w_3)) \vee C_1(w_1, w_2, w_3) \wedge (\neg S(w_1) \vee S(w_2) \vee S(w_3)) \vee$$

$$C_2(w_1, w_2, w_3) \wedge (\neg S(w_1) \vee \neg S(w_2) \vee S(w_3)) \vee C_3(w_1, w_2, w_3) \wedge (\neg S(w_1) \vee \neg S(w_2) \vee \neg S(w_3)).$$

• MAX SAT is a problem in the class MAX Σ_1 (cf. [PY88]). Under the encoding of SATISFIABILITY given in Section 2, if $\mathbf{A}(I)$ is the finite structure associated with an instance I of MAX SAT, then we have

$$opt(\mathbf{A}(I)) = \max_{S} |\{w : \mathbf{A}(I) \models (\exists y)[C(w) \wedge ((P(w, y) \wedge S(y)) \vee (N(w, y) \wedge \neg S(y)))]\}|.$$

have chosen to use different names for MAX SNP and MAX NP here, because we are interested in having a uniform notation and terminology for all the classes of optimization problems obtained using first-order formulae. Moreover, the notation Π_n and Σ_n is consistent with the notation Π_n^p and Σ_n^p used for the polynomial hierarchy [Sto76]. The class MAX Π_1 was introduced by Panconesi and Ranjan [PR90].

3 Polynomially Bounded NP Maximization Problems

In this section we investigate the relative expressive power of the classes MAX Π_n and MAX Σ_n, $n \geq 0$, and establish their basic relationship to the class MAX $\mathcal{PB}$ of polynomially bounded NP maximization problems.

Theorem 1: Let σ be a vocabulary and let $\mathcal{Q}$ be a maximization problem with finite structures $\mathbf{A}$ over σ as instances. Then $\mathcal{Q}$ is a polynomially bounded NP maximization problem if and only if there is a first-order formula $\phi(\mathbf{w}, \mathbf{S})$ with predicate symbols among those in σ and $\mathbf{S}$ such that for every instance $\mathbf{A}$ of $\mathcal{Q}$

$$opt_{\mathcal{Q}}(\mathbf{A}) = \max_{\mathbf{S}} |\{\mathbf{w} : \mathbf{A} \models \phi(\mathbf{w}, \mathbf{S})\}|.$$

Moreover, $\phi(\mathbf{w}, \mathbf{S})$ can always be taken to be a Π_2 formula and, consequently,

$$\text{MAX } \mathcal{PB} = \text{MAX } \Pi_2 = \text{MAX } \Pi_n, \ n > 2.$$

Proof: It is clear that if a maximization problem $\mathcal{Q}$ is in the class MAX Π_n for some $n \geq 0$, then $\mathcal{Q}$ is a polynomially bounded NP maximization problem, since for any finite structure $\mathbf{A}$ there are polynomially many distinct tuples from $\mathbf{A}$ satisfying a given first-order formula.

For the other direction, assume that $\mathcal{Q}$ is a polynomially bounded NP maximization problem with instances finite structures $\mathbf{A}$ over the vocabulary σ. Let m be a positive integer such that for any instance $\mathbf{A}$ we have that $\text{opt}_{\mathcal{Q}}(\mathbf{A}) \leq |\mathbf{A}|^m$, where $|\mathbf{A}|$ is the size of the structure $\mathbf{A}$.

Consider now the following decision problem Q: Given a finite structure $\mathbf{A}$ over σ and a m-ary relation W on the universe A of $\mathbf{A}$, is there a feasible solution T for $\mathbf{A}$ such that $f_{\mathcal{Q}}(\mathbf{A}, T) \geq |W|$? Here, $f_{\mathcal{Q}}$ is the objective function of $\mathcal{Q}$ and $|W|$ is the cardinality of the m-ary relation W. Since $\mathcal{Q}$ is an NP optimization problem, we have that Q is a problem in NP. Moreover, Q can be viewed as an NP decision problem with instances finite structures over the vocabulary $\sigma \cup \{W\}$. By Fagin's [Fag74] characterization of NP in terms of definability in second-order logic, there is an existential second-order formula $(\exists \mathbf{S}^*)\psi(\mathbf{S}^*, W)$ such that a pair $(\mathbf{A}, W)$ is a YES instance of Q if and only if $(\mathbf{A}, W) \models (\exists \mathbf{S}^*)\psi(\mathbf{S}^*, W)$. Since the maximization problem $\mathcal{Q}$ is bounded by $|\mathbf{A}|^m$, we have that

$$\text{opt}_{\mathcal{Q}}(\mathbf{A}) = \max_{\mathbf{S}^*, W}\{|W| : \mathbf{A} \models \psi(\mathbf{S}^*, W)\}$$

TRAVELING SALESMAN problem and INTEGER PROGRAMMING are examples of NP optimization problems that are not polynomially bounded.

Usually, NP decision problems can be represented as problems on finite structures over some vocabulary σ consisting of predicate symbols. Indeed, in most cases either an NP decision problem is described directly as a problem on finite structures or it can be easily encoded by such a problem. For example, CLIQUE and VERTEX COVER are problems about finite graphs, while an instance I of SATISFIABILITY can be identified with a finite structure $\mathbf{A}(I) = (X, C, P, N)$, where X is the set of variables and clauses of I, the predicate $C(x)$ expresses that x is a clause, and $P(c, v)$ and $N(c, v)$ are binary predicates expressing that a variable v occurs positively or negatively in a clause c.

From now on we assume that the instances of an optimization problem are given as finite structures over some vocabulary σ. We introduce next a framework for classifying optimization problems on finite structures in terms of their logical definability.

Recall that $\Sigma_n, n \geq 1$, is the class of first-order formulae in prenex normal form that have n alternations of quantifiers and start with a block of existential quantifiers. For example, Σ_1 is the collection of existential formulae, while Σ_2 is the class of existential-universal formulae. Similarly, Π_n, $n \geq 1$, is the class of first-order formulae in prenex normal form with n alternations of quantifiers, starting with a block of universal quantifiers. Thus, a Π_1 formula has universal quantifiers only, while Π_2 is the collection of universal-existential formulae. The class of quantifier-free formulae is denoted by Σ_0 or by Π_0.

Definition 2.3: Let σ be a vocabulary and let $\mathcal{Q}$ be a maximization problem with finite structures $\mathbf{A}$ over σ as instances.

We say that $\mathcal{Q}$ is in the class MAX Π_n, $n \geq 0$, if there is a Π_n formula $\phi(\mathbf{w}, \mathbf{S})$ with predicate symbols among those in σ and $\mathbf{S}$ such that for every instance $\mathbf{A}$ of $\mathcal{Q}$ we have that

$$opt_{\mathcal{Q}}(\mathbf{A}) = \max_{\mathbf{S}} |\{\mathbf{w} : \mathbf{A} \models \phi(\mathbf{w}, \mathbf{S})\}|.$$

Similarly, we say that $\mathcal{Q}$ is in the class MAX Σ_n, $n \geq 0$, if its optimum is definable as above using a Σ_n formula $\phi(\mathbf{w}, \mathbf{S})$.

The classes MIN Π_n and MIN Σ_n, $n \geq 0$, of minimization problems are defined in an analogous way, with min in place of max. In particular, a minimization problem $\mathcal{Q}$ is in the class MIN Π_n, $n \geq 0$, if there is a Π_n formula $\phi(\mathbf{w}, \mathbf{S})$ with predicate symbols among those in σ and $\mathbf{S}$ such that for every instance $\mathbf{A}$ of $\mathcal{Q}$ we have that

$$opt_{\mathcal{Q}}(\mathbf{A}) = \min_{\mathbf{S}} |\{\mathbf{w} : \mathbf{A} \models \phi(\mathbf{w}, \mathbf{S})\}|.$$

The classes MAX Σ_0 and MAX Σ_1 were introduced and studied by Papadimitriou and Yannakakis [PY88] under the names MAX SNP and MAX NP respectively. We in some sense a "dual" of the class RMAX in [PR90]. This subclass of MIN Σ_1 contains MIN VERTEX COVER and has the property that every minimization problem in it is approximable.

2 Preliminaries

This section contains the basic definitions and a minimum amount of the necessary background material.

Definition 2.1: An NP *optimization problem* is a tuple $\mathcal{Q} = (\mathcal{I}_\mathcal{Q}, \mathcal{F}_\mathcal{Q}, f_\mathcal{Q}, opt)$ such that

- $\mathcal{I}_\mathcal{Q}$ is the set of input instances. It is assumed that $\mathcal{I}_\mathcal{Q}$ can be recognized in polynomial time.
- $\mathcal{F}_\mathcal{Q}(I)$ is the set of feasible solutions for the input I.
- $f_\mathcal{Q}$ is a polynomial time computable function, called the *objective function*. It takes positive integer values and is defined on pairs (I, T), where I is an input instance and T is a feasible solution of I.
- $opt \in \{\max, \min\}$
- The following decision problem is in NP : Given $I \in \mathcal{I}_\mathcal{Q}$ and an integer k, does there exist a feasible solution $T \in \mathcal{F}_\mathcal{Q}(\mathcal{I})$ such that $f_\mathcal{Q}(I, T) \geq k$, when $opt =$ max? (or, $f_\mathcal{Q}(I, T) \leq k$, when $opt = \min$)

The above definition is due to [PR90] and is broad enough to encompass every known optimization problem arising in NP-completeness. We now restrict attention to *polynomially bounded* NP optimization problems [BJY89,LM81]. These are NP optimization problems in which the optimum value of the objective function on an instance is bounded by a polynomial in the length of that instance.

Definition 2.2: An NP optimization problem $\mathcal{Q}$ is said to be *polynomially bounded* if there is a polynomial p such that

$$opt(I) \leq p(|I|) \text{ for all } I \in \mathcal{I}_\mathcal{Q}.$$

Let MAX $\mathcal{PB}$ (MIN $\mathcal{PB}$) be the set of all polynomially bounded NP maximization (minimization) problems.

Examples of polynomially bounded NP optimization problems are MAX CLIQUE, TRAVELING SALESMAN problem with weights 1 or 2, MIN COLORING, and MIN VERTEX COVER. On the other hand, the unrestricted version of the optimum value is less than or equal to a polynomial of the input size. We classify next these problems according to the quantifier complexity of the first-order formulae used and we show that they form a proper hierarchy with exactly four levels:

$$\text{MAX } \Sigma_0 \subset \text{MAX } \Sigma_1 \subset \text{MAX } \Pi_1 \subset \text{MAX } \Pi_2,$$

where MAX Σ_0 = MAX SNP is obtained using quantifier-free formulae, MAX Σ_1 = MAX NP is obtained using existential formulae, MAX Π_1 is obtained using universal formulae, and finally MAX Π_2 is obtained using universal-existential formulae. In particular, MAX Π_2 can capture every polynomially bounded NP-maximization problem on finite structures. The above containments are strict and there are natural maximization problems witnessing the separation of the four classes. We prove that MAX CONNECTED COMPONENT is in MAX Π_2, but not in MAX Π_1, while MAX SAT separates MAX Σ_1 from MAX Σ_0. As mentioned above, [PR90] showed that MAX CLIQUE is in MAX Π_1, but not in MAX Σ_1.

We focus next on the logical definability of NP-minimization problems. Panconesi and Ranjan [PR90] concentrated on maximization problems only, while Papadimitriou and Yannakakis [PY88] examined approximation properties of certain minimization problems by reducing them to maximization problems. At first sight, one may expect that results about classes of maximization problems should translate directly to analogous results about classes of minimization problems definable by similar formulae. It turns out, however, that this is *not* the case. Actually, maximization and minimization problems defined by similar first-order formulae may have strikingly different approximation properties.

We show that the collection of polynomially bounded NP-minimization problems on finite structures coincides with the class of minimization problems whose optimum is defined using an existential-universal (Σ_2) first-order formula. After this we establish that the polynomially bounded NP-minimization problems can be classified into a proper hierarchy with exactly three levels:

$$\text{MIN } \Sigma_0 \subset \text{MIN } \Sigma_1 \subset \text{MIN } \Pi_1 = \text{MIN } \Sigma_2.$$

The above containments are strict. In fact, we show that MIN CHROMATIC NUMBER is in MIN Π_1, but not in MIN Σ_1, while MIN VERTEX COVER is in MIN Σ_1, but does not belong to MIN Σ_0.

Recall that Papadimitriou and Yannakakis [PY88] showed that every maximization problem in MAX Σ_0 = MAX SNP or in MAX Σ_1 = MAX NP is approximable. In contrast, we prove here that MIN Σ_0 contains natural minimization problems, such as MIN 3NON-TAUTOLOGY, that are not approximable, unless P=NP. Since the quantifier pattern of minimization problems does not have an impact on the approximation properties of the problems, we seek other syntactic properties that may have such an impact. To this effect, we introduce a natural subclass of MIN Σ_1 that is corresponding maximization problem in MAX NP one seeks predicates $\mathbf{S}$ that maximize the number of tuples $\mathbf{x}$ satisfying the existential first-order sentence $(\exists \mathbf{y})\psi(\mathbf{x}, \mathbf{y}, \mathbf{S})$. MAX SAT is the canonical example of a problem in MAX NP. This problem asks for the maximum number of clauses that can be satisfied in a given Boolean formula.

Papadimitriou and Yannakakis [PY88] showed that every optimization problem in MAX NP can be approximated within a constant factor. They also considered the subclass MAX SNP of MAX NP consisting of those maximization problems that are defined by quantifier-free formulae, i.e., the optimum of such problems can be defined as

$$\max_{\mathbf{S}} |\{\mathbf{x} : \mathbf{A} \models \psi(\mathbf{x}, \mathbf{S})\}|,$$

where ψ is quantifier-free. They demonstrated that MAX SNP contains several natural maximization problems that are complete for MAX SNP via a certain reduction that preserves approximability. MAX 3SAT is a typical MAX SNP-complete problem. These results on the one hand reveal that the logical definability of an optimization problem may impact on its approximation properties and on the other provide an explanation as to why polynomial-time approximation schemes have not been derived for MAX 3SAT or for the other MAX SNP-complete problems.

More recently, Panconesi and Ranjan [PR90] investigated the expressive power of MAX NP and showed that MAX CLIQUE does not belong to this class. Moreover, they proved that certain polynomial-time optimization problems are not in MAX NP. In an attempt to find a syntactic class of optimization problems containing MAX CLIQUE, they introduced the class MAX Π_1 of maximization problems whose optimum can be defined as

$$\max_{\mathbf{S}} |\{\mathbf{w} : \mathbf{A} \models (\forall \mathbf{x})\psi(\mathbf{w}, \mathbf{x}, \mathbf{S})\}|,$$

where ψ is quantifier-free. It turns out that MAX Π_1 contains also maximization problems that are not approximable within a constant, unless P=NP. In view of this, Panconesi and Ranjan [PR90] studied the class RMAX, which is a syntactic subclass of MAX Π_1 containing MAX CLIQUE and having the property that every problem in it is *self-improvable*.

What other classes of optimization problems can be obtained using the logical definability perspective and what is the exact expressive power of this framework?

We address these questions here by examining the class of all maximization problems whose optimum is definable using first-order formulae, i.e., it is given as

$$\max_{\mathbf{S}} |\{\mathbf{w} : \mathbf{A} \models \phi(\mathbf{w}, \mathbf{S})\}|,$$

where $\phi(\mathbf{w}, \mathbf{S})$ is an arbitrary first-order formula. We show first that this class coincides with the collection of *polynomially bounded* NP-maximization problems on finite structures, namely, the NP-maximization problems on finite structures whose

1 Introduction and Summary of Results

It is well known that optimization problems had a major influence on the development of the theory of NP-completeness. As a matter of fact, many natural NP-complete problems are decision problems that are derived from an optimization problem by imposing a bound on the objective function ([GJ79]). In spite of this close connection, NP-completeness advanced along a strikingly different path than that of optimization theory. Non-deterministic Turing machines with polynomial-time bounds provide a fairly robust computational model for decision problems. This, in turn, made it possible to develop a rich structural complexity theory based on polynomial time reductions and to obtain various classifications of NP problems. There have been also several attempts to classify optimization problems and to study their structural properties. Some notable contributions include [OM90,Kre88,Wag86,PM81,ADP80,Joh74] (cf. also [BJY89] for a comprehensive survey of results in this area). Nevertheless, the absence of robust computational models for optimization problems has hindered the development of a structural optimization theory that is on a par with structural complexity theory. In particular, the approximation properties of optimization problems remain as one of the most persistent puzzles of optimization theory. Although all known natural NP-complete problems are polynomially isomorphic [BH77], their optimization counterparts may have dramatically different approximation properties, from possessing polynomial-time approximation schemes to being non-appproximable within a constant factor (assuming P$\neq$NP).

Papadimitriou and Yannakakis [PY88] brought a fresh perspective to approximation theory by focusing on the logical definability of optimization problems. Their main motivation came from Fagin's [Fag74] characterization of NP in terms of definability in second-order logic on finite structures. An *existential second-order formula* is an expression of the form $(\exists \mathbf{S})\phi(\mathbf{S})$, where $\mathbf{S}$ is a sequence of predicates and $\phi(\mathbf{S})$ is a first-order formula. Fagin's theorem [Fag74] asserts that a collection C of finite structures is NP-computable if and only if it is definable by an existential second-order formula. Moreover, it is well known that every such formula is equivalent to one of the form $(\exists \mathbf{S})(\forall \mathbf{x})(\exists \mathbf{y})\psi(\mathbf{x},\mathbf{y},\mathbf{S})$, where ψ is a quantifier-free formula. Thus, a class C of finite structures is NP-computable if and only if there is a formula $(\exists \mathbf{S})(\forall \mathbf{x})(\exists \mathbf{y})\psi(\mathbf{x},\mathbf{y},\mathbf{S})$, with ψ quantifier-free, such that for any finite structure $\mathbf{A}$ we have that

$$\mathbf{A} \in C \iff \mathbf{A} \models (\exists \mathbf{S})(\forall \mathbf{x})(\exists \mathbf{y})\psi(\mathbf{x},\mathbf{y},\mathbf{S}).$$

Papadimitriou and Yannakakis [PY88] introduced the class MAX NP of maximization problems whose optimum can be defined as

$$\max_{\mathbf{S}} |\{\mathbf{x} : \mathbf{A} \models (\exists \mathbf{y})\psi(\mathbf{x},\mathbf{y},\mathbf{S})\}|,$$

where ψ is quantifier-free. Intuitively, in an NP decision problem one seeks predicates $\mathbf{S}$ witnessing some existential second-order sentence $(\exists \mathbf{S})(\forall \mathbf{x})(\exists \mathbf{y})\psi(\mathbf{x},\mathbf{y},\mathbf{S})$, while in the

Logical Definability of NP Optimization Problems *

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Abstract: We investigate here NP optimization problems from a logical definability standpoint. We show that the class of optimization problems whose optimum is definable using first-order formulae coincides with the class of polynomially bounded NP optimization problems on finite structures. After this, we analyze the relative expressive power of various classes of optimization problems that arise in this framework. Some of our results show that logical definability has different implications for NP maximization problems than it has for NP minimization problems, in terms of both expressive power and approximation properties.

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