Multivariate Estimation of the Properties of an Oil Reservoir

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October 4, 2007

Abstract

We consider the problem of joint estimation of some of the soil properties of an oil reservoir, like porosity and sand thickness. In an exploration scenario, only a few wells have been drilled, thus data from the wells are scarce. In contrast, there is an abundance of seismic data. In our example, which corresponds to a Venezuelan oil reservoir, the data available from the wells consist of gamma ray logs measured as a function of depth. The seismic data correspond to traces obtained around the wells. The average properties of the soil corresponding to a given range of depths for the wells are known from direct measurement. The goal is to predict those properties at points where only seismic data are available. We fit a multivariate linear regression model that accounts for the spatial correlation using spatial kernels. These are based on the family of Matèrn correlations. The kernels provide weighting of the information in the signals that are dependent on spatial locations. We first transform the dependent variable using discrete wavelets. We then perform a Bayesian variable selection procedure using a Metropolis search. This allows for the detection of the most informative wavelet coefficients. Not all the soil properties are available at all wells, so we use a Bayesian approach to handle the missing data. We obtain predictions of all the properties over the whole reservoir. Thanks to the Bayesian nature of our method we are able to provide a probabilistic quantification of the predictive uncertainties. The cross-validated results show that very high accuracy can be achieved even with a very small number of wavelet coefficients.

1 Introduction

Predicting the properties of the soil in a reservoir is a fundamental activity during the initial stages of oil exploration. The basic objective of all geophysical exploration is to

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employ the principles of physics, especially the physical properties of rocks, to determine the disposition of rocks below the surface of the earth. Oil does not accumulate in all types of rock, hydrocarbons may travel large distances throughout the porous medium until they find the proper conditions that help to trap them. A rock possessing both porosity and permeability must be available to receive the hydrocarbons from their source sediments; a rock lacking those properties must be adjacent to the porous and permeable rock to prevent the fluids from escaping. For this reason, determining properties of rocks in a certain oil exploration area becomes a fundamental task for efficient exploitation of existing reserves (see for example, Tiab and Donaldson, 2004; Hearst and Nelson, 2000; Schon, 2006).

The most common sources of quantitative information consist of: Core analyses, performed on actual samplings of the soil from the wells; Recordings of different electromagnetic, physical, chemical or radioactive characteristics of the soil, obtained by inserting various tools into the wells. These are usually referred to as well logs or well profiles; Seismic traces recorded over a dense grid of points throughout the reservoir. Core data are usually scarce, very localized and expensive to obtain. Several well logs are routinely obtained from the wells for various depths. The most common profiles are those of electric, radioactive and electromagnetic measures.

An example of a profile used to measure radioactivity is the γ-ray log. The γ-ray log is a measurement of the natural radiation of various formations penetrated by a well (or in some cases artificially placed sources of radiation). It can be recorded in open and cased holes, separately or in conjunction with virtually any other log, or with perforating guns. Dolomites, limestones, sandstones and salts typically exhibit a low level of radiation, while shales, clays, and rocks of igneous origin typically have higher levels of radiation. These differences make the γ-ray log very useful in determining lithologies and in the evaluation of the shale volume, porosity and other rock properties in zones of interest, (see for example, Robinson, 2000; Hearst and Nelson, 2000). This will be the well log used in the present paper.

Seismic traces are obtained from an impulse or electric wave generated at the surface. A vibratory source sends waves into the ground. These waves travel through the ground and are recorded by devices located on a regular grid at the surface. Their traveling speed depends on the density and rigidity of the medium, so seismic waves are affected by the characteristics of the rock that they penetrate. They provide an indirect measurement of the properties of a reservoir.

Figure 1 shows the profiles of a γ-ray well log and a seismic trace available for the reservoir that we consider in this paper. We notice that the y axis of the two panels have different units. Well logs are measured at different depths and seismic signals are recorded as a time series. Accurate matching of the recording times with the corresponding depths requires precise knowledge of the soil properties. An initial mapping of the two scales is usually done by experts using geological information, (see for example, Lisle, 2003)

The data considered in this paper were recorded on a reservoir located between the States of Apure and Barinas in Western Venezuela. We have γ-ray logs for 14 wells and seismic data covering an area of about 100 km^2. A reservoir is a three dimensional domain. Here we focus in the estimation of the average clay volume, potassium level and porosity in a window of 150 feet, starting at 9,000 feet of depth. So we tackle a two dimensional prediction problem.

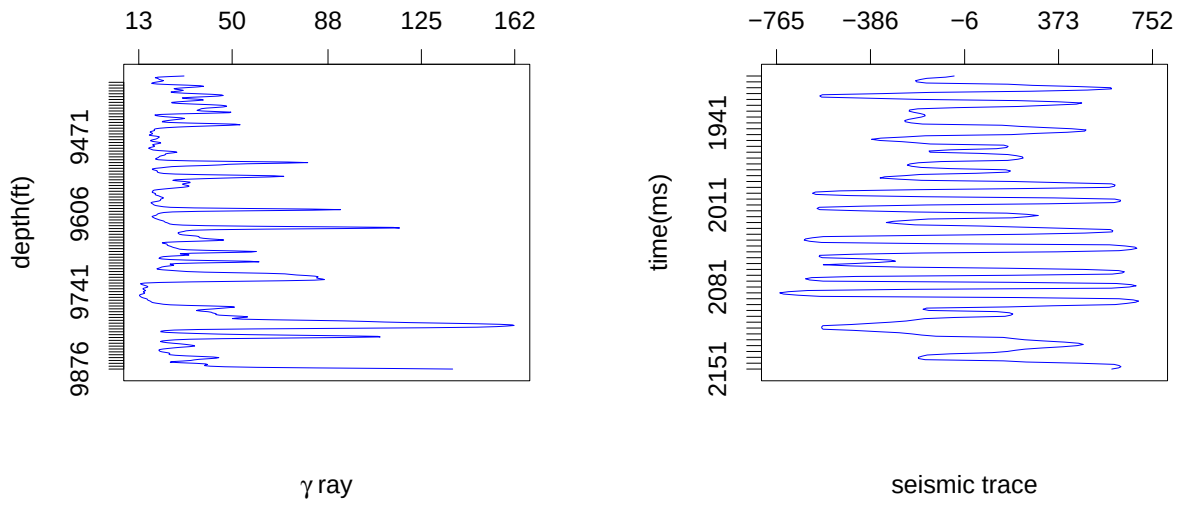

Figure 1: Left panel shows a typical γ-ray log for the wells in the reservoir. Right panel corresponds to a typical seismic trace.

This segment of the reservoir has been identified as being of petrophysical interest. It is considered to be reasonably homogeneous and it has been estimated to correspond to 32 seconds of seismic traces. The locations of the wells are irregularly scattered. They are presented in Figure 2. We notice that all three properties are available at all 14 wells, with the exception of potassium levels, which is missing at wells 1,8 and 13. The goal of this work is to obtain predictive fields for all three properties over the whole area covered by the seismic data.

From the petroleum engineering point of view the problem considered in this paper is challenging due to the fact that the reservoir is in the exploration phase. In contrast to a production scenario, where information from hundreds of wells may be available, only few wells are drilled for exploration. Predicting the properties of a reservoir is usually done using geostatistical methods like kriging and co-kriging (see for example Cressie, 1993; Chambers et al., 2002; Sheldon, 1995). Such methods can have very low accuracy when only few spatial locations are available. Additionally, seismic information, can be difficult to incorporate in an effective way. For a critical review of the use of geostatistical methods for the exploration of oil reservoirs see Hirshe and Porter-Hirsche (1997).

This paper addresses a number of statistical issues. The proposed approach uses information from different sources to produce inference about a multivariate random field on a two dimensional space. We do so by using well logs and seismic traces as explanatory variables. We transform them using wavelets. This creates an overparameterized problem that is dealt with using stochastic variable selection. The results show that about 2% of the wavelet coefficients of the well logs and 9% of those of the seismic traces are needed for accurate predictions. This provides a useful summary of the information, which consists of 289,926 seismic traces, amounting to 566 Mb of data. The spatial dependence between locations is

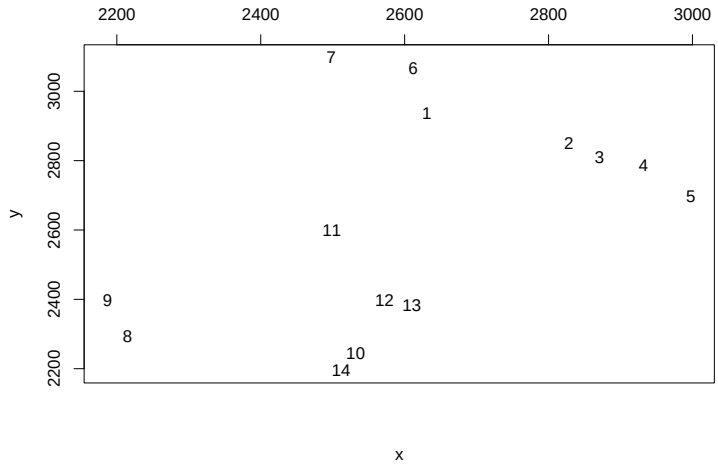

Figure 2: Locations of the 14 well in the reservoir under study. Wells are numbered so that they can be referenced in the analysis.

modeled using kernels centered around the locations of the existing wells.

We use a Bayesian approach that allows us to use prior distributions that informs the variable selection process. It also incorporates all estimation uncertainties in the prediction and quantifies them using probabilities. The results produced by the model are compatible with the lithological composition of the reservoir as inferred by geophysicists with good knowledge of the area. Recent references on the prediction of the properties of an oil reservoir using Bayesian methods are Eidsvik et al. (2004); Larsen et al. (2006); Eide et al. (2002).

In Section 2 we develop the model and discuss the parameter estimation, soil properties prediction and model assessment. In Section 4 we present the results and finally, in Section 5 we discuss our findings. We have included three appendixes that are helpful for a better understanding of the proposed model and estimation method.

2 A multivariate model for soil properties

We consider a matrix of response values consisting of 14 rows, one for each well location, and three columns, one for each of the soil properties. We assume that the well logs and the seismic traces have some predictive capability for the properties. Thus, at each point of the surface for which a seismic record is available, we regress the responses on the seismic traces. Additionally, we include weighted averages of the regressed values of the well logs. The weights are dependent on the distance between the given location and that of the well logs. So, the 'soft' information provided by the seismic traces is only used locally. The 'hard' information from the wells is used globally but spatially weighted. Álvarez and Sansó (2007) found that wavelet transforms of γ-ray well logs as well as those of seismic traces measured around the well are good predictors of porosity and sand thickness. Moreover, less than 2% of the coefficients are enough to produce accurate predictions. This is not surprising

since the multiresolution properties of wavelets have been successfully applied to quantify the decay of energy from large to small scales in well logs and seismic traces (see for example Álvarez et al., 2003).

Let $\boldsymbol{Y} \in \mathbb{R}^{14\times 3}$ be the matrix of response values. We consider the regression model

$$\begin{aligned} \boldsymbol{Y} &= \mathbb{1}\boldsymbol{\alpha}_s' + \boldsymbol{X}_s\boldsymbol{B}_s + \boldsymbol{M} + \boldsymbol{E}_s \\ \boldsymbol{M} &= \boldsymbol{K}\mathrm{vec}(\boldsymbol{\mu}) \\ \boldsymbol{\mu} &= \mathbb{1}\boldsymbol{\alpha}_w' + \boldsymbol{X}_w\boldsymbol{B}_w + \boldsymbol{E}_w \ . \end{aligned} \tag{1}$$

Here $\boldsymbol{\alpha}_s \in I\!R^q$ and $\boldsymbol{\alpha}_w \in I\!R^q$ are the intercepts, with $q = 3$. Letting $n = 14$, $\boldsymbol{X}_s \in I\!R^{n\times p_s}$ and $\boldsymbol{X}_w \in I\!R^{n\times p_w}$ are the matrices of seismic traces and well logs respectively. We assume that columns of $\boldsymbol{X}_w$ and $\boldsymbol{X}_s$ have been centered. Since each seismic trace has 128 observations and each well log has 512, we have that $p_s = 128$ and $p_w = 512$. $\boldsymbol{B}_s \in I\!R^{p_s\times q}$ and $\boldsymbol{B}_w \in I\!R^{p_w\times q}$ are the corresponding matrices of the regression coefficients. $\boldsymbol{E}_s$ and $\boldsymbol{E}_w$ are noise matrices of dimension $n \times q$. $\boldsymbol{\mu} \in I\!R^{n\times q}$ is a matrix of latent parameters providing the link between the information at the wells and the soil properties at a given point. $\mathrm{vec}(\cdot)$ operates on a matrix by stacking its columns into one vector.

$\boldsymbol{K} \in I\!R^{nq\times nq}$ is a matrix of kernel values calculated as a function of the distance between points in the surface of the reservoir. More specifically $\boldsymbol{K}$ is a block diagonal matrix with q blocks. Each block is of dimensions $n \times n$ and corresponds to one of the columns in $\boldsymbol{Y}$. Let $\boldsymbol{u}_i$ be the location of the i-th well, then the elements of the j-th block are given by

$$k_{it}^{(j)} = \frac{l_{it}^{(j)}}{l_i} \quad t = 1, \ldots, n \quad i = 1, \cdots, n \quad j = 1, \ldots, q.$$

where

$$l_{it}^{(j)} = \left(\frac{||\boldsymbol{u}_i - \boldsymbol{u}_t||}{\lambda_j}\right)^{\nu} K_\nu\left(\frac{||\boldsymbol{u}_i - \boldsymbol{u}_t||}{\lambda_j}\right); \ \ l_i = \sum_{k=1}^{n} l_{it}^{(j)}. \tag{2}$$

$K_\nu(\cdot)$ is the modified Bessel function of the second kind of order $\nu > 0$ (see Abramowits and Stegun, 1970) and $\lambda_j > 0, j = 1, \ldots, q$. The use of the kernel defined in (2) is motivated by the class of Matèrn correlation functions (see for example, Stein, 1999). In fact, a Gaussian process with a given correlation function, can be represented under fairly general conditions as the convolution of white noise with an appropriate kernel (see for example, Higdon, 2002). Processes with correlation functions in the Matèrn class can be represented using kernels of the type in Equation (2). Please refer to Appendix A for details. Thus the parameters λ_j provide information about the correlation range and ν measures the smoothness of the resulting random field. Here we are assuming that ν is common to all q responses, but that each component has a different range parameter. Given that we only evaluate the kernels at 14 points, the model in Equation (1) gives only a crude approximation to the continuous process convolution. Nevertheless, it provides an effective and parsimonious way of capturing the spatial structure of the data.

To specify the distribution of the error matrices $\boldsymbol{E}_s$ and $\boldsymbol{E}_w$ we use a matrix normal distribution. See Appendix B for notation and properties. Thus,

$$\boldsymbol{Y} - \mathbb{1}\boldsymbol{\alpha}_s' - \boldsymbol{X}_s\boldsymbol{B}_s - \boldsymbol{M} \sim \mathcal{MN}(\boldsymbol{I}, \boldsymbol{\Sigma}_s)$$

$$\mu - \mathbb{1}\boldsymbol{\alpha}_w' - \boldsymbol{X}_w\boldsymbol{B}_w \sim \mathcal{NM}(\boldsymbol{I}, \boldsymbol{\Sigma}_w),$$

Where $\boldsymbol{I}$ denotes the identity matrix. So, in both cases, the rows are independent but the columns have a non-diagonal covariance matrix. This implies that, after accounting for the information in the signals and conditioning on all parameters, the responses from different wells do not have any correlation, but there is a correlation among the different properties that does not depend on location.

Notice that the model has a larger number of regression coefficients than data. Thus direct estimation of $\boldsymbol{B}_s$ and $\boldsymbol{B}_w$ using traditional regression methods is unfeasible. We either have to impose some restrictions or consider prior information. Conditional on $\boldsymbol{K}$, we can obtain a conjugate prior for $\boldsymbol{\alpha}_s, \boldsymbol{\alpha}_s, \boldsymbol{B}_s, \boldsymbol{B}_w, \boldsymbol{\Sigma}_s$ and $\boldsymbol{\Sigma}_w$ by using a normal-Wishart distribution. This is obtained by assuming that, conditional on $\boldsymbol{\Sigma}_s$ and $\boldsymbol{\Sigma}_w$, $\boldsymbol{\alpha}_i' - \boldsymbol{\alpha}_{i_0}' \sim \mathcal{NM}(\boldsymbol{I}h_i, \boldsymbol{\Sigma}_i)$ and $\boldsymbol{B}_i \sim \mathcal{NM}(\boldsymbol{H}_i, \boldsymbol{\Sigma}_i)$, where $i = s$ or w. $h_i \in \mathbb{R}$ and the matrices $\boldsymbol{\alpha}_{i_0}$ and $\boldsymbol{H}_i$ are assumed known. The Distributions of $\boldsymbol{\Sigma_s}$ and $\boldsymbol{\Sigma_w}$ are Inverse Wishart (see Appendix B for Definition), and we write $\boldsymbol{\Sigma}_i \sim \mathcal{IW}(\delta, \boldsymbol{Q}_i), i = s, w$.

Motivated by the work of Brown et al. (2001) we consider a wavelet transformation of the signals. The idea of such a transformation is that only a reduced number of wavelet coefficients are needed to predict the value of the properties at a given location. A discrete wavelet transformation is given by an orthogonal matrix (see for example, Vidakovic, 1999), say $\boldsymbol{W} \in \mathbb{R}^{p\times p}$, such that $\boldsymbol{W}\boldsymbol{W}' = \boldsymbol{I}$. We then have that

$$\boldsymbol{Y} - \mathbb{1}\boldsymbol{\alpha}_s' - \boldsymbol{Z}_s\boldsymbol{\beta}_s - \boldsymbol{M} \sim \mathcal{NM}(\boldsymbol{I}, \boldsymbol{\Sigma}_s) \tag{3}$$

where $\boldsymbol{Z}_s = \boldsymbol{X}_s\boldsymbol{W}$ is the matrix of wavelet coefficients for the seismic traces and $\boldsymbol{\beta}_s = \boldsymbol{W}'\boldsymbol{B}_s$. A transformation similar to the above one yields, for the well profiles, $\boldsymbol{\mu} - \mathbb{1}\boldsymbol{\alpha}_w' - \boldsymbol{Z}_w\boldsymbol{\beta}_w \sim \mathcal{NM}(\boldsymbol{I}, \boldsymbol{\Sigma}_w)$, thus $\boldsymbol{\beta}_s$ and $\boldsymbol{\beta}_w$ are the new matrices of regression coefficients.

From Equation (3) we obtain the likelihood

$$f(\boldsymbol{Y}|\boldsymbol{\alpha}_s, \boldsymbol{\beta}_s, \boldsymbol{\Sigma}_s, \boldsymbol{\mu}, \boldsymbol{\lambda}, \nu) \propto |\boldsymbol{\Sigma}_s|^{-n/2} \exp\left\{-\frac{1}{2}\text{tr}\left[(\boldsymbol{Y} - \mathbb{1}'\boldsymbol{\alpha}_s - \boldsymbol{Z}_s\boldsymbol{\beta}_s - \boldsymbol{M})\boldsymbol{\Sigma}_s{}^{-1}(\boldsymbol{Y} - \mathbb{1}'\boldsymbol{\alpha}_s - \boldsymbol{Z}_s\boldsymbol{\beta}_s - \boldsymbol{M})'\right]\right\}$$

and

$$f(\boldsymbol{\mu}|\boldsymbol{\alpha}_w, \boldsymbol{\beta}_w, \boldsymbol{\Sigma}_w) \propto |\boldsymbol{\Sigma}_w|^{-n/2} \exp\left\{-\frac{1}{2}\text{tr}\left[(\boldsymbol{\mu} - \mathbb{1}'\boldsymbol{\alpha}_w - \boldsymbol{Z}_w\boldsymbol{\beta}_w)\boldsymbol{\Sigma}_w{}^{-1}(\boldsymbol{\mu} - \mathbb{1}'\boldsymbol{\alpha}_w - \boldsymbol{Z}_w\boldsymbol{\beta}_w)'\right]\right\}.$$

The prior for the parameters $\boldsymbol{\alpha}_s$, $\boldsymbol{\alpha}_w$, $\boldsymbol{\Sigma_s}$, $\boldsymbol{\Sigma_w}$, are unchanged by the orthogonal transformations and we can write their prior distributions as

$$\begin{aligned}
\pi(\boldsymbol{\alpha}_i|\boldsymbol{\Sigma}_i) &\propto h^{-q/2}|\boldsymbol{\Sigma}_i|^{-1/2}\exp\left[-\frac{1}{2h}(\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i_0})\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i_0})'\right] \\
\pi(\boldsymbol{\Sigma}_i|\delta, \boldsymbol{Q}) &\propto |\boldsymbol{\Sigma}_i|^{-(\delta/2+q)}\exp\left[\frac{1}{2}\text{tr}\left(\boldsymbol{Q}_i\boldsymbol{\Sigma}_i^{-1}\right)\right] \quad i = s, w.
\end{aligned}$$

The prior for the transformed regression matrices $\boldsymbol{\beta}_w$ and $\boldsymbol{\beta}_s$ are

$$\pi(\boldsymbol{\beta}_i|\boldsymbol{\Sigma}_i) \propto |\tilde{\boldsymbol{H}}_i|^{-q/2}|\boldsymbol{\Sigma_i}|^{-p/2} \exp\left[-\frac{1}{2}\text{tr}\left(\tilde{\boldsymbol{H}}_i^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\beta}_{i_0})\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\beta}_{i_0})'\right)\right], \quad i = s, w,$$

where $\tilde{\boldsymbol{H}}_i = \boldsymbol{W}\boldsymbol{H}_i\boldsymbol{W}'$. In order to compute these matrices, we use the the recursive algorithm developed in Vannucci and Corradi (1999), which speeds up computations by using two-dimensional discrete wavelet transforms.

2.1 Selection of wavelet coefficients

To perform the selection of wavelet coefficients we use a stochastic variable selection approach, as in Brown et al. (1998). We define latent binary vectors $\boldsymbol{\gamma}^w$ and $\boldsymbol{\gamma}^s$ of dimension p_w and p_s respectively. Each component of $\boldsymbol{\gamma}^i, i = s, w$ denotes the presence or absence of the corresponding coefficient. To incorporate this information into the prior for $\boldsymbol{\beta}_w$ and $\boldsymbol{\beta}_s$ we assume that the covariance matrix of a given row of $\boldsymbol{\beta}_i$ is zero when the corresponding component of $\boldsymbol{\gamma}^i$ is zero. Denote $\boldsymbol{\beta}_i^{\gamma}$ the matrices of non-zero regression coefficients, then we have that $\boldsymbol{\beta}_i^{\gamma} \sim \mathcal{NM}(\tilde{\boldsymbol{H}}_i^{\gamma}, \boldsymbol{\Sigma}_i)$, where $\tilde{\boldsymbol{H}}_i^{\gamma}$ is equal to the appropriate sub-matrix of $\tilde{\boldsymbol{H}}_i$. Denote as $\boldsymbol{\beta}_i[j:]$ the j-th column of $\boldsymbol{\beta}_i$, then under the proposed prior

$$\boldsymbol{\beta}_i[j:] \sim (1 - \gamma_j^i)\boldsymbol{I}_0 + \gamma_j^i \mathcal{N}(0, \tilde{h}_{jj}^i \boldsymbol{\Sigma}_i), \quad i = w, s,$$

where $\tilde{h}_{jj}^i$ corresponds to the j-th element of the diagonal of the matrices $\tilde{\boldsymbol{H}}_i$ and $\boldsymbol{I}_0$ is a point mass at 0. We note that, if $\boldsymbol{\Sigma}_i$ is not diagonal, the rows of $\boldsymbol{\beta}_i$ are not independent.

To complete the prior we need to specify the distribution for γ_j^i and values for $\boldsymbol{H}_i, \delta$ and $\boldsymbol{Q}_i$. We assume that $\gamma_j^i \sim Bernoulli(\omega_i),\ j = 1, 2 \cdots, p_i,\ i = w, s$ for some ω_i. Experience shows that good predictions can be obtained with about 20 wavelet coefficients. Thus our prior distribution for γ_s and γ_w is such that the prior expected number of coefficients, $p\omega$, is equal to 20. Thus $\omega = 0.16$ for the seismic traces and $\omega = 0.04$ for logs. The inverse Wishart prior for Σ_i implies that $E(\boldsymbol{\Sigma}_i) = \boldsymbol{Q}_i/(\delta - 2)$, if $\delta \geq 3$. So we choose $\delta = 3$ and set $\boldsymbol{Q}_s = 0.06\boldsymbol{I}_q$ and $\boldsymbol{Q}_w = 0.1\boldsymbol{I}_q$. So that, a priori, we have no expected off-diagonal terms and the expected scales are comparable to the error variances of the standardized Y given X.

To set values of $\boldsymbol{H_i}$, $i = w$,s we performed a preliminary analysis using partial least squares regression on each of the three soil properties separately. We observed that the resulting coefficients are reasonably smooth. Thus, we expect the regressors $\boldsymbol{B_i}$ to be reasonably smooth as well. Such smoothness is imposed by taking $\boldsymbol{H_i}$ to be the covariance matrix of a first-order autoregressive process. Thus, we assume that $h_{jl}^i = {\sigma_i}^2 \rho_i^{|j-l|}$, for some σ_i and ρ_i, $i = w, s$. We follow Brown et al. (1998) for the derivation of such constants. Using an empirical Bayes approach we set $\sigma_w^2 = 351$, $\rho_w = 0.26$ and $\sigma_s^2 = 298$ and $\rho_s = 0.14$ by maximizing the marginal density of $\boldsymbol{Y}$. We observe that the resulting variances of the transformed coefficients $\boldsymbol{\beta}_s$ and $\boldsymbol{\beta}_w$ show the typical exponential decay of wavelet coefficients.

3 Estimation and prediction

To obtain inferences on the parameters in our model we need to explore a probability distribution on a very highly dimensional space. This can be done by using a Markov chain

Monte Carlo method (MCMC) as proposed, for example, in Gamerman and Lopes (2006). We use a traditional MCMC that consists of sampling iteratively from the distributions of each of the parameters or blocks of parameters conditional on all the remaining ones. $\boldsymbol{\alpha}_s$ and $\boldsymbol{\alpha}_w$ can be sampled from q-dimensional multivariate normal distributions, but in the present application we set them to zero after centering $\boldsymbol{Y}, \boldsymbol{X}_s$ and $\boldsymbol{X}_w$. We sample vec$(\boldsymbol{\mu})$ from a nq-dimensional multivariate normal. $\boldsymbol{\beta}_s^{\boldsymbol{\gamma}}$ and $\boldsymbol{\beta}_w^{\boldsymbol{\gamma}}$ are sampled from matrix-normal distributions of dimensions that are appropriate for the number of non-zero entries in $\boldsymbol{\gamma}^i, i = s, w$. $\boldsymbol{\Sigma}_s$ and $\boldsymbol{\Sigma}_w$ are sampled from $q \times q$-dimensional inverse Wishart distributions. See Appendix C for explicit expressions for the parameters of the full conditional distributions.

Generating samples of $\boldsymbol{\gamma}^s$ and $\boldsymbol{\gamma}^w$ presents the challenge of dealing with a highly multivariate distribution, since in our case p_s is 128 and p_w is 512. The use of Metropolis search to find configurations with high posterior probabilities has been developed in Madigan and York (1995) and applied by George and McCulloch (1993), Raftery et al. (1997) and Brown et al. (2001). We proceed by considering a random initial configuration. Then, at each iteration, one of the following two ways of choosing a candidate configuration is chosen with (fixed) probability ϕ: (a) Generate a new candidate by choosing at random a component. This component is deleted if it is part of the current configuration and added if it is not; (b) Select two components k and j such that $\gamma_k^i = 0$ and $\gamma_j^i = 1$ and swap their values. The proposed configuration is rejected or accepted following a Metropolis-Hastings rule.

The acceptance probability for the new value γ^* is

$$\alpha(\boldsymbol{\gamma}, \boldsymbol{\gamma}^*) = \min\left\{1, \frac{q(\boldsymbol{\gamma}^*)}{q(\boldsymbol{\gamma})}\right\}.$$

Finally, the parameters in the kernels matrix $\boldsymbol{K}$, $\lambda_1, \lambda_2, \lambda_3$ and ν are sampled using Metropolis-Hastings. We use a gamma-prior distribution for the kernels parameters.

3.1 Predictive Distribution

As stated in the Introduction the goal of our model is to estimate the soil properties at locations of the reservoir where no wells have been drilled and only seismic information is available. Denote such values as $\tilde{\boldsymbol{Y}}$. Then we base our estimates on the posterior predictive distribution of $\tilde{\boldsymbol{Y}}$ given $\boldsymbol{Y}, \boldsymbol{X}_s$ and $\boldsymbol{X}_w$. Bayesian computations in a predictive problem are based on the joint posterior distribution of parameters and variables to be predicted. Denote as $\boldsymbol{\theta}$ the set of all parameters in the model, then posterior predictive distribution of $\tilde{\boldsymbol{Y}}$ is

$$P(\tilde{\boldsymbol{Y}}|\boldsymbol{Y}, \boldsymbol{X}_s, \boldsymbol{X}_w) = \int P(\tilde{\boldsymbol{Y}}|\boldsymbol{X}_s, \boldsymbol{X}_w, \boldsymbol{\theta}) P(\boldsymbol{\theta}|\boldsymbol{Y}, \boldsymbol{X}_s, \boldsymbol{X}_w) d\boldsymbol{\theta},$$

since $P(\tilde{\boldsymbol{Y}}|\boldsymbol{Y}, \boldsymbol{X}_s, \boldsymbol{X}_w, \boldsymbol{\theta}) = P(\tilde{\boldsymbol{Y}}|\boldsymbol{X}_s, \boldsymbol{X}_w, \boldsymbol{\theta})$. This is due to the fact that the spatial dependence between properties at different locations is explained by $\boldsymbol{M} = \boldsymbol{K}vec(\boldsymbol{\mu})$, and these quantities are given.

Suppose that the output from the MCMC consists of L blocks of values $\boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(L)}$. Consider a location u with corresponding (unobserved) soil properties $\tilde{\boldsymbol{Y}}(u)$, then we ap-

proximate the predictive posterior distribution of $\tilde{\boldsymbol{Y}}(u)$ as

$$\hat{P}(\tilde{\boldsymbol{Y}}(u)|\boldsymbol{Y}, \boldsymbol{X}_s, \boldsymbol{X}_w) = \frac{1}{L}\sum_{j=1}^{L} P(\tilde{\boldsymbol{Y}}(u)|\boldsymbol{\theta}^{(j)}, \boldsymbol{X}_s, \boldsymbol{X}_w)$$

where

$$P(\tilde{\boldsymbol{Y}}(u)|\boldsymbol{\theta}^{(j)}, \boldsymbol{X}_s, \boldsymbol{X}_w) = \mathcal{N}(\mathbb{1}'\boldsymbol{\alpha}_s^{(j)} + \boldsymbol{Z}_s^*\boldsymbol{\beta}_s^{(j)} + \boldsymbol{m}^*, \boldsymbol{\Sigma}_s^{(j)}).$$

Here $\boldsymbol{m}^* = (m_1^*, m_2^*, m_3^*)$ and $m_i^* = \sum_{l=1}^{n} k_i(\boldsymbol{u} - \boldsymbol{w}_l)\mu_{li}^{(j)}$, $\boldsymbol{w}_l$ is the location of the l-th well. $\boldsymbol{Z}_s^*$, corresponds to the wavelet transformation of the signal at location $\boldsymbol{u}$. We notice that $\boldsymbol{\gamma}^s$ and $\boldsymbol{\gamma}^w$ are two of the components of $\boldsymbol{\theta}$ and that they may vary from one iteration to the next. So the number of regression coefficients may change as the MCMC evolves, defining potentially different models. Our predictions are based on averaging across all models visited by the Markov chain.

4 Results

We fitted the model using wavelet transformations based on the Haar basis. Haar wavelet basis are a popular choice in geological applications due its simplicity and its ability to detect sudden changes in the signals. We present results that were obtained from 5,000 iterations of a MCMC after a burn in period of 500 iterations. To establish the convergence of the chain we use the method in Raftery and Lewis (1992) with the default values provided in `BOA` package of `R` (Smith, 2005; R Development Core Team, 2005). This yields a minimum number of 3,740 iterations with a dependence factor smaller than 3.

Table 1 shows the quantiles of the kernel parameters. Regarding the estimation of Σ_s and Σ_w we have their posterior means are

$$\begin{pmatrix} 1.635 & -0.0225 & -0.035 \\ -0.0225 & 0.942 & 0.046 \\ -0.035 & 0.046 & 1.26 \end{pmatrix} \text{ and } \begin{pmatrix} 2.0520 & -0.091 & -0.086 \\ -0.0091 & 1.1512 & 0.071 \\ -0.086 & 0.071 & 1.12 \end{pmatrix}$$

respectively. This suggests that the first column in both errors are negatively correlated with the other two, while the second and the third are positively correlated.Table 2 shows the 2.5%, 50% and 97.5% posterior quantiles for the correlations $c_{ij}^{(w)} = \Sigma_{ij}^{(w)}/\sqrt{\Sigma_{ii}^{(w)}\Sigma_{jj}^{(w)}}$, and $c_{ij}^{(s)} = \Sigma_{ij}^{(s)}/\sqrt{\Sigma_{ii}^{(s)}\Sigma_{jj}^{(s)}}$ $i, j = 1, 2, 3$. The numbers indicate that the correlation are not significant.

As mentioned in the previous section, the number of non-zero coefficients can vary from one iteration of the MCMC to the next. Nevertheless we observed that no more than 10 coefficients were different from zero at any given iteration. This implies that with less than 2% of the wavelet coefficient is possible to obtain accurate predictions. Figure 3 shows the values predicted for the three soil properties over the whole reservoir. Figure 4 shows the estimated interquartile ranges for the properties of the reservoir over the whole area. We observe greater predictive uncertainty in the zones without wells. This is expected, since in those areas prediction depend mainly on the seismic data.

Parameters	2.5%	50%	97.5%
λ_1	0.5	0.69	0.75
λ_2	1.28	1.42	1.49
λ_3	0.67	0.72	0.77
ν	0.93	1.7	2.35

Table 1: Posterior quantiles of the Kernel parameters

quantiles	$c_{12}^{(w)}$	$c_{13}^{(w)}$	$c_{23}^{(w)}$	$c_{12}^{(s)}$	$c_{13}^{(s)}$	$c_{23}^{(s)}$
2.5%	-0.1043	-0.1743	-0.1632	-0.1916	-0.1678	-0.1287
50%	-0.0181	-0.0243	0.0422	-0.0592	-0.0569	0.0546
97.5%	0.2215	0.1249	0.2312	0.2714	0.0643	0.1014

Table 2: Posterior quantiles of the Correlation Matrices

To explore the predictive capability of the model we adopted a "leave one out" validation approach, consisting on obtaining the posterior predictive distribution for each of the 14 locations using the remaining 13. The predictions for each well are given by the medians of the simulated values obtained from the MCMC samples. In Figure 5 we compare the interquartile ranges of the posterior predictive distributions of each of the 14 wells, based on the remaining 13 wells, to the actual observed values of porosity, sand thickness and potassium levels. We observe that in all cases the observations are within the predicted range. A more detailed predictive assessment is presented in Figures 6 and 7. We selected Well 11, which is central to the domain, and estimated the joint posterior predictive density for the soil properties using the MCMC samples. The comparison with the observed values show that these are very central to the predictive densities in both the univariate and bivariate cases. The former shows that the method has a very high level of predictive accuracy. Also, given the Bayesian nature of the method, we are not only providing an estimate of the properties at each location, but a precise assessment of the uncertainties involved in such estimation, given by the predictive distribution.

5 Discussion and Conclusions

We have presented a method to estimate jointly several petrophysical properties of an oil reservoir during the exploration phase, using abundant seismic traces and scarce well information. The methodology is able to produce an interpolated field over the whole reservoir with a probabilistic assessment of the uncertainties in the estimations. The cross-validation of the predictions yields accurate results. The use of wavelet transformations results in an effective way of summarizing the information in the two different types of signals. In fact, it is seen that only a small proportion of the wavelet coefficients are needed to obtain realistic estimations of the soil properties. This points at the possibility of using wavelets methods for data reduction purposes in this type of analysis. The use of a suitable structured prior for the wavelet coefficients is key to achieve a parsimonious estimation in an otherwise overparameterized problem. The spatial structure of the data is described using kernel convolutions.

These offer a valid alternative to other spatial inference methods, as they provide a link between spatially indexed signals of different dimensions, available at different locations.

Additional information regarding the structure of the reservoir could be incorporated in the analysis by embedding the current model within a hierarchical structure. In an exploration scenario, such information is likely to consist of geological knowledge about the subsurface. A clear limitation for its use is that fact that it may not be completely independent from the signals already considered in this paper.

Acknowledgments

We are indebted to Reinaldo Michelena, PhD, formerly at INTEVEP, the R&D of Petroleos de Venezuela S.A., who proposed the problem and provided the data to us. We acknowledge the help of Professor Marina Vannucci who made some matlab routines available to us. The second author was partially funded by the National Science Foundation grant DMS 0504851.

A Process convolutions

A process convolution is given by

$$z(s) = \int_S k_\psi(u - s) dW(u) \tag{4}$$

for $s \in S$, S a subset of an Euclidean space, W a Wiener process and $k_\psi(\cdot)$ a kernel, possibly depending on a low dimensional parameter ψ. The covariance of the process $z(s)$ defined in Equation (4) is given by the convolution of the kernel with itself, so

$$c(s, s') = \mathrm{cov}(z(s), z(s')) = \int_S k_\psi(u - d) k_\psi(u) du \ \ ,$$

where $d = s - s'$. If $k_\psi(\cdot)$ depends only on the magnitude of d, then so does the covariance function $c(\cdot)$, implying that $z(s)$ is isotropic. If the kernel is square-integrable, for a given $c(\cdot)$, $k_\psi(\cdot)$ can be obtained as the inverse Fourier transform of the square root of the spectral density of c. For the Matèrn class of isotropic correlations in $\mathbb{R}^2$, with range $\lambda > 0$ and smoothness $\nu > 0$, the spectral density is given by $f(\omega) \propto 1/(\lambda^2 + \omega^2)^{\nu+1}$. The corresponding kernel is the inverse Fourier transform of $1/(\lambda^2 + \omega^2)^{\nu/2+1/2}$, which is proportional to

$$(\lambda s)^{\nu/2-1/2} \mathcal{K}_{\nu-1/2}(\lambda s), \quad \lambda > 0, \nu > 1.$$

In practice a discrete version of Equation (4) is used.

B Matrix-variate distributions

We follow the notation used by Dawid (1981) for matrix-variate distributions. This has the advantage of preserving matrix structures.

Definition 1 *: We say that L has a matrix-variate distribution, $\boldsymbol{X} - \boldsymbol{M} \sim \mathcal{NM}(\boldsymbol{\Gamma}, \boldsymbol{\Sigma})$, if $L = M + AUB$, with U a matrix having independent standard normal entries, M, A and B fixed matrices satisfying $A'A = \Gamma$, $B'B = \Sigma$. Thus M is the matrix mean of V,and $\gamma_{ii}\Sigma$ and $\sigma_{jj}\Gamma$ are the covariance matrices of the ith row an jth columns respectively of L.*

Form the above definition we obtain a procedure for the generation of matrix normal samples based on the Cholesky decomposition of Γ and Σ.

Definition 2 *We say that W has an inverse Wishart distribution with scale matrix Σ and shape parameter δ if $W = B'(U'U)^{-1}B$, with U and L defined as above. We use as notation: $W \sim IW(\delta, \Sigma)$. The Expectation of W exists for $\delta > 2$ and is then $\Sigma/(\delta - 2)$.*

Definition 3 *Let $\mathbf{A} = a_{ij}$ be a $p_1 \times p_2$ matrix and let $\mathbf{B} = b_{ij}$ be a $q_1 \times q_2$ matrix. Then the Kronecker Product of $\mathbf{A}$ and $\mathbf{B}$ is the $p_1 q_1 \times p_2 q_2$ matrix $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$,*

$$\begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1p_2}\mathbf{B} \\ \vdots & \vdots & & \vdots \\ a_{p_1 1}\mathbf{B} & a_{p_1 2}\mathbf{B} & \cdots & a_{p_1 p_2}\mathbf{B} \end{pmatrix}$$

Lemma 1 *Let be $\boldsymbol{X} \in I\!R^{(m \times n)}$ and denote its columns as $x_i, i = 1, \ldots, n$. Assume that $\boldsymbol{X} - \boldsymbol{M} \sim \mathcal{NM}(\boldsymbol{\Gamma}, \boldsymbol{\Sigma})$ and let*

$$\text{vec}(\boldsymbol{X}) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then $\text{vec}(\boldsymbol{X}) \sim \mathcal{NM}(\text{vec}(\boldsymbol{M}), \boldsymbol{\Sigma} \otimes \boldsymbol{\Gamma})$, where $\boldsymbol{\Sigma} \otimes \boldsymbol{\Gamma} \in I\!R^{nq \times nq}$ is the Kronecker Product of $\boldsymbol{\Gamma}$ y $\boldsymbol{\Sigma}$.

Lemma 2 *Let be $\boldsymbol{Y} = \boldsymbol{BXZ}, \boldsymbol{Y} - \boldsymbol{M} \sim \mathcal{NM}(\boldsymbol{\Gamma}, \boldsymbol{\Sigma})$, $\boldsymbol{B} \in I\!R^{r \times m}$ and $\boldsymbol{C} \in I\!R^{n \times c}$ constant matrices and $\text{vec}(\boldsymbol{Y}) = \boldsymbol{C} \otimes \boldsymbol{B}\ \text{vec}(\boldsymbol{X})$, then*

$$\begin{aligned} E(\text{vec}(\boldsymbol{Y})) &= (\boldsymbol{C}' \otimes \boldsymbol{B})\ \text{vec}(\boldsymbol{M}) \\ cov(\text{vec}(\boldsymbol{Y})) &= (\boldsymbol{C}'\boldsymbol{\Gamma}\boldsymbol{C}) \otimes (\boldsymbol{B}\boldsymbol{\Sigma}\boldsymbol{B}) \end{aligned}$$

C Full Conditional Distributions

- $\boldsymbol{\beta}_i,\ i = s, w$:

$$\pi(\boldsymbol{\beta}_i|\cdot) \propto \exp\left[-\frac{1}{2} tr\left(\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\Lambda}_{\boldsymbol{\beta}_i}\boldsymbol{b}_i)\boldsymbol{\Lambda}_{\boldsymbol{\beta}_i}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\Lambda}_{\boldsymbol{\beta}_i}\boldsymbol{b}_i)'\right)\right]$$

  where $\boldsymbol{\Lambda}_{\boldsymbol{\beta}_i}^{-1} = \boldsymbol{Z}_i{}'\boldsymbol{Z}_i + \boldsymbol{H}_i^{-1}$ and $\boldsymbol{b}_i = \boldsymbol{Z}_i(\boldsymbol{Y} - \mathbb{1}'\boldsymbol{\alpha}_i - \boldsymbol{M})$.

- $\boldsymbol{\Sigma}_i,\ i = s, w$:

$$\pi(\boldsymbol{\Sigma_i}|\cdot) \propto |\boldsymbol{\Sigma}_i|^{-(\delta^*/2+q)} \exp\left[-\frac{1}{2} tr \boldsymbol{Q}^* \boldsymbol{\Sigma}_i^{-1}\right]$$

where $\delta^* = n + p_i + 1 + \delta$, $\boldsymbol{Q}_i^* = c_1 + c_2 + c_3$ $c_1 = (\boldsymbol{\beta}_i - \boldsymbol{\beta}_{i_0})\boldsymbol{H}_i(\boldsymbol{\beta}_i - \boldsymbol{\beta}_{i_0})', c_2 = (\boldsymbol{Y} - \mathbb{1}'\boldsymbol{\alpha}_i - \boldsymbol{Z}_i\boldsymbol{B}_i - M)(\boldsymbol{Y} - \mathbb{1}'\boldsymbol{\alpha}_i - \boldsymbol{Z}_i\boldsymbol{B}_i - M)'$ and $c_3 = \boldsymbol{Q}_i$.

- $\boldsymbol{\mu}$:

$$\pi(\boldsymbol{\mu}|\cdot) \propto \exp\left[-\frac{1}{2}(\boldsymbol{\mu} - \boldsymbol{m})'\boldsymbol{\Lambda}_{\boldsymbol{\mu}}^{-1}(\boldsymbol{\mu} - \boldsymbol{m})\right]$$

where $\boldsymbol{m} = \boldsymbol{\Lambda\mu}\left[\boldsymbol{K}'\left(\boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_s\right)^{-1} vec(\boldsymbol{Y}) + \left(\boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_w\right)^{-1} vec(\boldsymbol{\mu}_w)\right]$, $\boldsymbol{\Lambda}_{\boldsymbol{\mu}} = \boldsymbol{K}'\left(\boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_s\right)^{-1}\boldsymbol{K} + \left(\boldsymbol{I}_n \otimes \boldsymbol{\Sigma}_w\right)^{-1}$ and $\boldsymbol{\mu}_w = vec(\mathbb{1}'\boldsymbol{\alpha}_w - \boldsymbol{Z}_w\boldsymbol{\beta}_w)$.

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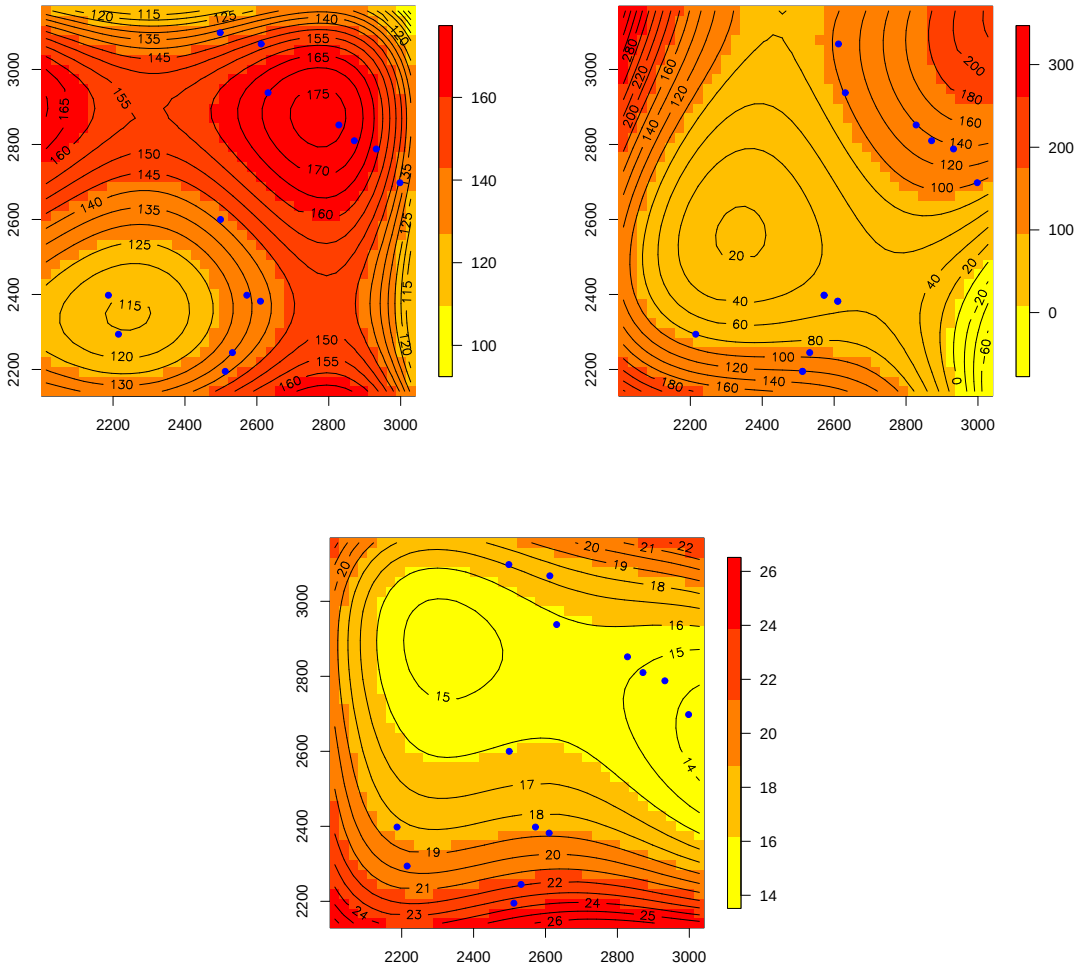

Figure 3: Estimated median soil properties of the reservoir. Upper left panel corresponds to clay volume. Upper right panel corresponds to Potassium levels. Lower panel corresponds to porosity.

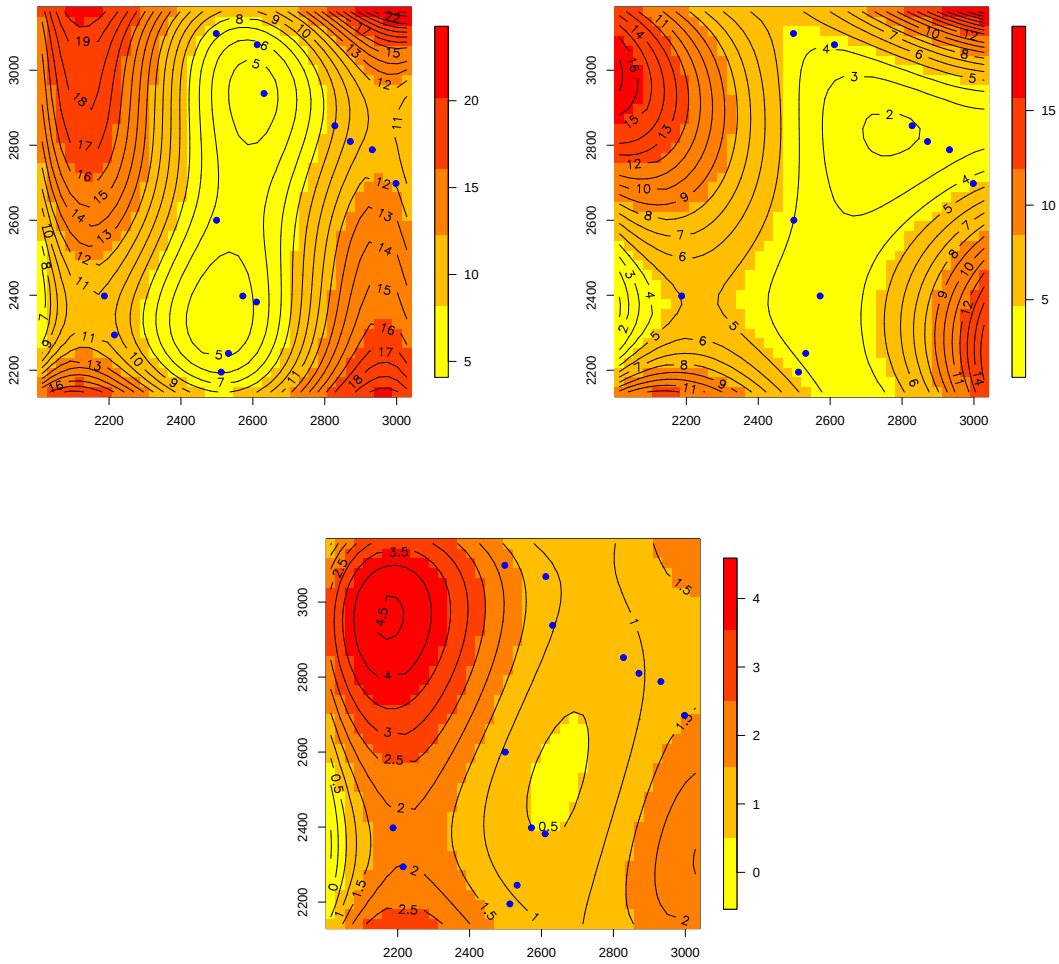

Figure 4: Estimated interquartile ranges for the soil properties of the reservoir. Upper left panel corresponds to clay volume. Upper right panel corresponds to Potassium levels. Lower panel corresponds to porosity.

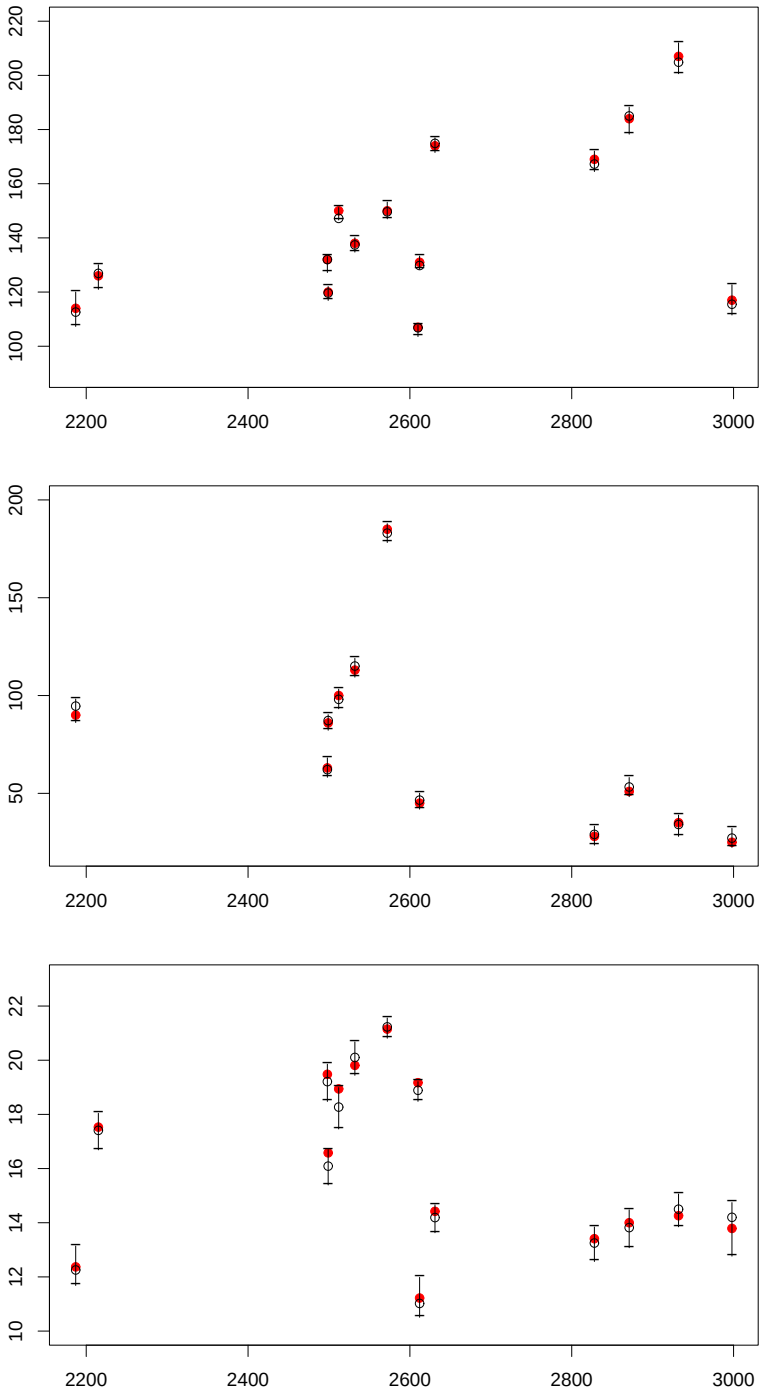

Figure 5: Posterior predictive interquartile intervals of each of the 14 wells obtained using the remaining 13 wells. The wells are ordered with respect to the x coordinate. Top panel corresponds to clay volume, center panel to Potassium levels and bottom panel to porosity. Actual observations marked as a solid circle. Posterior predictive median marked as a circle.

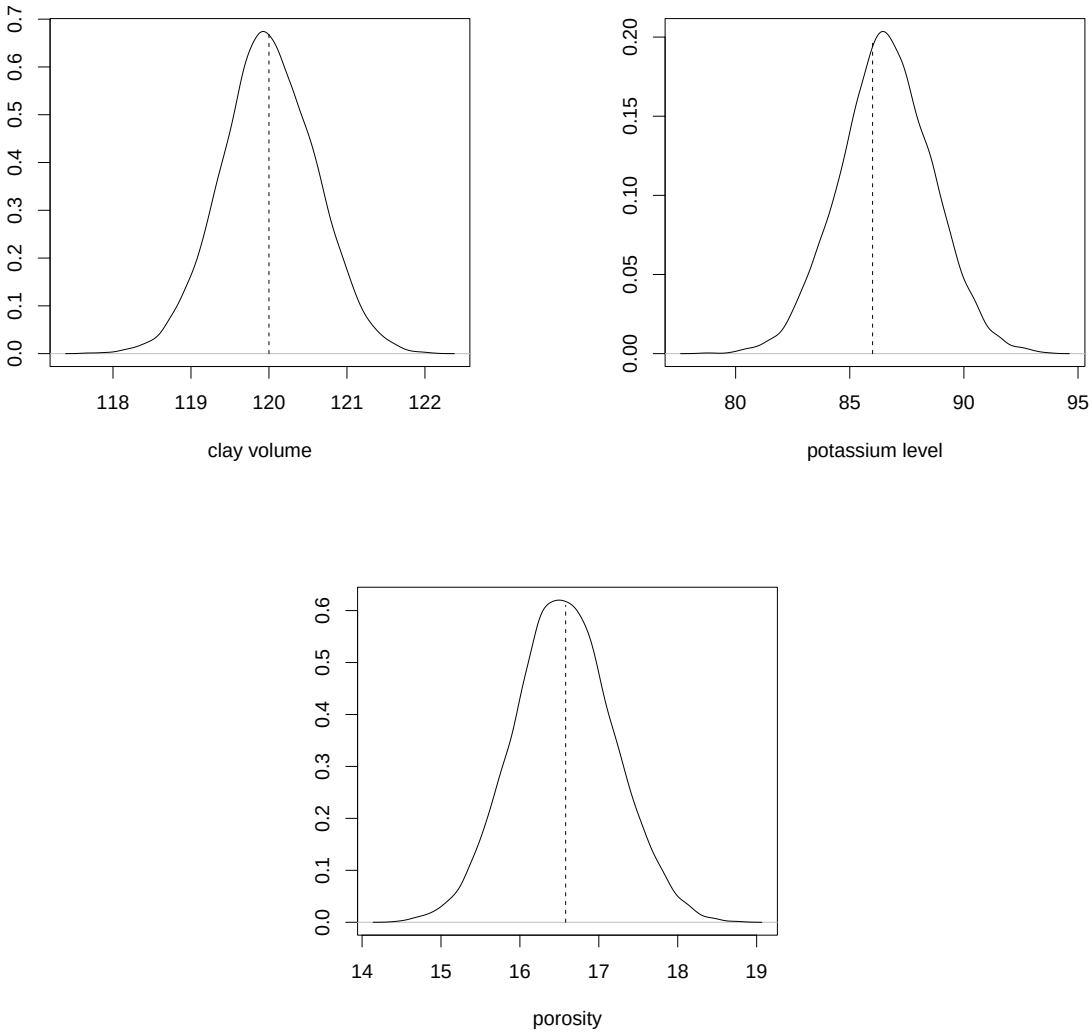

Figure 6: Univariate marginal predictive densities for the soil properties at Well 11. Clay volume (bottom left), potassium level (bottom right) and porosity (down). The dotted lines correspond to the observed values.

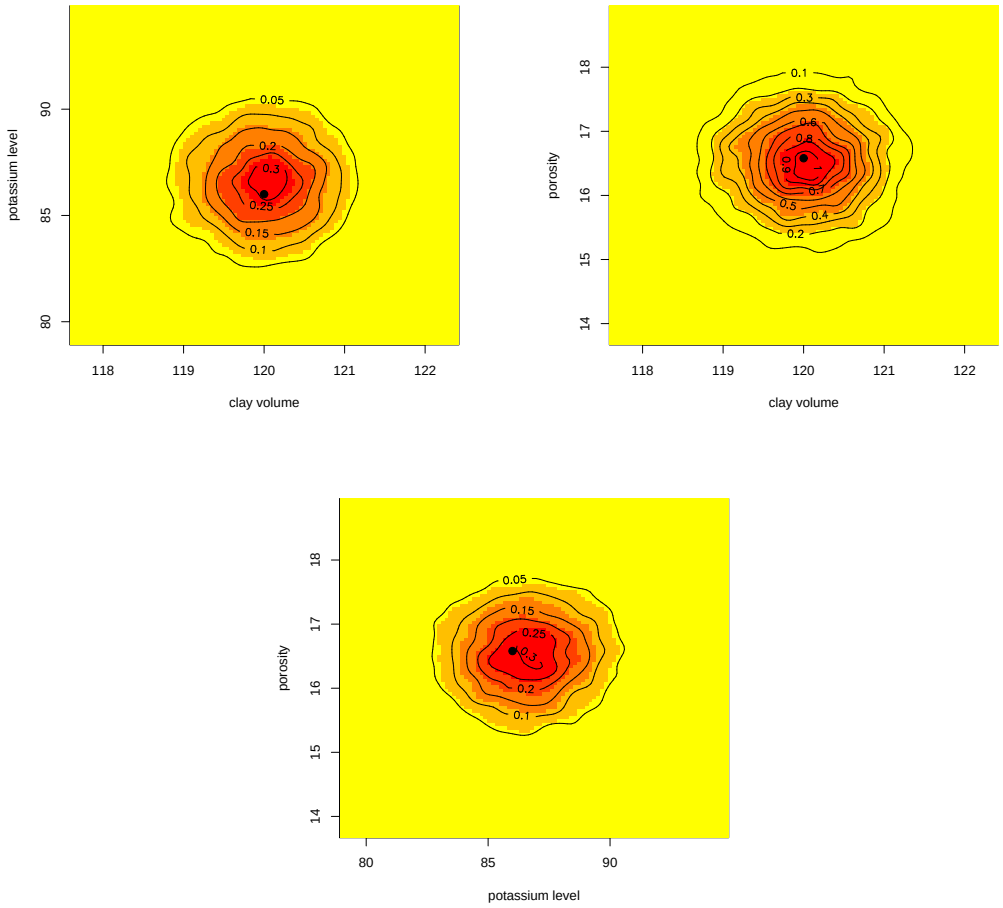

Figure 7: Bivariate marginal predictive densities for the soil properties at Well 11. The black dots correspond to the observed values.