A practical model of convective dynamics for stellar evolution calculations

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Abstract. Turbulent motions in the interior of a star play an important role in its evolution, since they transport chemical species, thermal energy and angular momentum. Our overall goal is to construct a practical turbulent closure model for convective transport that can be used in a multi-dimensional stellar evolution calculation including the effects of rotation, shear and magnetic fields. Here, we focus on the first step of this task: capturing the well-known transition from radiative heat transport to turbulent convection with and without rotation, as well as the asymptotic relationship between turbulent and radiative transport in the limit of large Rayleigh number. We extend the closure model developed by Ogilvie (2003) and Garaud and Ogilvie (2005) to include heat transport and compare it with experimental results of Rayleigh-Benard convection.

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INTRODUCTION

Turbulent convection plays an essential role in the evolution of most stars. Turbulent motions are often the dominant mechanism of energy transport in stellar convective zones, and can increase momentum transport and chemical mixing by several orders of magnitude. In many rotating astronomical systems where turbulence is anisotropic, Reynolds stresses are the dominant transporters of angular momentum and therefore influence the internal dynamics of the whole system.

It is currently not possible to perform a 3-D numerical simulation of convective motions over the evolution timescale of a star. However, when considering stellar evolution we are not necessarily interested in the specific details of the convective motions, but rather in statistical properties such as the convective flux and Reynolds stresses. Our long-term goal is to construct a closed set of evolution equations for these statistical quantities in terms of large-scale system properties (e.g. viscosity, rotation, shear), which can be used in a stellar evolution calculation.

In this paper we focus on modelling how the convective Reynolds stresses and heat fluxes are affected by rotation. We are especially interested in approximately predicting the onset of convection as well as the asymptotic behaviour of turbulent transport for large Rayleigh number; capturing the onset is required for the stellar evolution model to correctly place the boundary between the convective and radiative zones. Adequate prediction of the asymptotic behaviour of the convective turbulence is important to describe the amount of energy and angular momentum that is transported in the majority of the convective zone.

To quantify the quality of the proposed closure model we test it against linear stability analysis, numerical simulations and laboratory experiments of the rotating Rayleigh-Benard problem.

ROTATING RAYLEIGH BENARD CONVECTION

The typical Rayleigh-Benard convection setup is as follows: two rigid, ideally infinite, horizontal plates separated by a distance D confine a weakly compressible fluid between them. A temperature difference ΔT is maintained between a hot bottom and a cool top. The system is assumed to be rotating with average angular velocity $\underline{\Omega} = \Omega\hat{z}$ with gravity $\mathbf{g} = -g\hat{z}$. The Boussinesq approximation is valid in this system. The governing equations are

$$\begin{aligned} \partial_i u_i &= 0, & (1)\\ (\partial_t + u_k\partial_k)u_i + 2\varepsilon_{ijk}\Omega_j u_k &= -\alpha\Theta g_i - \partial_i\Psi + \nu\partial_{kk}u_i, & (2)\\ (\partial_t + u_k\partial_k)\Theta &= \kappa\partial_{kk}\Theta & (3) \end{aligned}$$

where the dynamical variables are the temperature offset Θ, the pressure perturbation from hydrostatic equilibrium Ψ and the flow velocity $\mathbf{u}$. The following parameters are assumed to be constant: the coefficient of expansion α, the kinetic viscosity ν and the thermal diffusivity κ. Sums over repeated indices are implied.

The qualitative behaviour of the system is controlled by three dimensionless quantities: the Rayleigh number, $\mathrm{Ra} \equiv \alpha g\Delta T D^3/(\kappa\nu)$ (measuring the ratio between buoyancy forces to viscous stabilising forces), the Taylor number, $\mathrm{Ta} \equiv 4D^4\Omega^2/\nu^2$ (measuring the ratio between centrifugal forces to viscous forces), and the Prandtl number, $\mathrm{Pr} \equiv \nu/\kappa$ (measuring the ratio between the viscous diffusion rate and the thermal diffusion rate). For any given Taylor number and Prandtl number, there exists a critical Rayleigh number ($\mathrm{Ra_c}$) above which the system is convective and below which the system is conductive. We strive to construct our model to match the known variation of the critical Rayleigh number with Taylor number and Prandtl number. We also verify that at Rayleigh numbers much larger than critical the Nusselt number Nu (ratio of total heat flux to conductive heat flux) is correctly predicted by the model.

CLOSURE MODEL

Our goal is to capture the behaviour of turbulent convection in a rotating system both in terms of the onset of turbulence and of its asymptotic properties. We developed a second order closure model using the technique described by Ogilvie (2003) and Garaud and Ogilvie (2005).

We write each quantity as the sum of a mean and fluctuating part ($u = \bar{u} + u'$, $\Psi = \bar{\Psi} + \Psi'$, and $\Theta = \bar{\Theta} + \Theta'$) – in the Rayleigh-Benard problem, these mean quantities only vary with z.

We define the following correlation quantities: $R_{ij} = \overline{u'_i u'_j}$, $F_i = \overline{\Theta' u'_i}$, and $Q = \overline{\Theta'\Theta'}$, so that $R = R_{ii}$ is twice the mean turbulent kinetic energy.

The exact equations governing the mean quantities are

$$\partial_i \bar{u}_i = 0 \tag{4}$$

$$(\partial_t + \bar{u}_k \partial_k)\bar{u}_i + 2\varepsilon_{ijk}\Omega_j \bar{u}_k = -\alpha\bar{\Theta}g_i - \partial_i \bar{\Psi} + \nu\partial_{kk}\bar{u}_i - \partial_j R_{ij} \tag{5}$$

$$(\partial_t + \bar{u}_k \partial_k)\bar{\Theta} = \kappa\partial_{ii}\bar{\Theta} - \partial_k F_k \tag{6}$$

while those governing the second order correlation terms are modelled as

$$\begin{aligned}(\partial_t + \bar{u}_k\partial_k)R_{ij} + R_{ik}\partial_k\bar{u}_j + R_{jk}\partial_k\bar{u}_i + 2\varepsilon_{ilm}\Omega_l R_{jm} + 2\varepsilon_{jlm}\Omega_l R_{im} + \alpha(F_i g_j + F_j g_i) \\ -\nu\partial_{kk}R_{ij} = -C_1\tau^{-1}R_{ij} - C_2\tau^{-1}(R_{ij} - \tfrac{1}{3}R\delta_{ij}) - \nu C_\nu L^{-2}R_{ij},\end{aligned} \tag{7}$$

$$\begin{aligned}(\partial_t + \bar{u}_j\partial_j)F_i + R_{ij}\partial_j\bar{\Theta} + F_j\partial_j\bar{u}_i + 2\varepsilon_{ijk}\Omega_j F_k + \alpha Q g_i - \frac{1}{2}(\nu+\kappa)\partial_{kk}F_i \\ = -C_6\tau^{-1}F_i - \frac{1}{2}(\nu+\kappa)C_\nu F_i L^{-2}, \text{ and}\end{aligned} \tag{8}$$

$$(\partial_t + \bar{u}_i\partial_i)Q + 2F_i\partial_i\bar{\Theta} - \kappa\partial_{kk}Q = -C_7\tau^{-1}Q - \kappa C_\nu Q L^{-2} \tag{9}$$

where the left-hand-side of each equation is exact, while the right-hand-side models the effect of higher order correlations. The constants $C_1, C_2, C_\nu, C_6, C_7$ are free parameters of the closure model[1]. The variable τ is the characteristic timescale for the redistribution of energy along the turbulent cascade, which is controlled by the turnover time of the largest eddies $d^{-1}R^{1/2}$ and L is the characteristic size of the perturbations near onset, which we now describe in more detail.

Following Prandtl's mixing length theory, Garaud and Ogilvie (2005) originally suggested that in a wall-bounded experiment L can be thought of as the distance to the wall. However, rotation does no net work on the system when Ω is parallel to $\mathbf{g}$ and since the closure model is constructed on energetic arguments, using this lengthscale prescription here fails to capture the known effects of rotation on the onset and turbulent properties of convection (Chandrasekhar (1961)).

Physically, rotation influences the onset of convection by decreasing the characteristic lengthscale of convective motions in the direction perpendicular to the rotation axis. To capture this effect, we construct our lengthscale L to be the harmonic mean between the distance to the wall and the wavelength of the most linearly unstable mode λ which, in the Rayleigh-Benard experiment, is a function of the Taylor number: therefore

$$L = \left(\frac{1}{d^2} + \frac{2}{\lambda^2}\right)^{-1/2}. \tag{10}$$

RESULTS

We seek solutions to equations (4) - (10) assuming no-slip boundary conditions and fixed uniform plate temperatures. The no-slip boundary condition states that $u_i = 0$ at

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[1] Note that $C_1 \simeq 0.4, C_2 \simeq 0.6$, and $C_\nu \simeq 12$ have already been found to give an adequate description of the turbulent stresses in Couette-Taylor experiments by Garaud and Ogilvie (2005), but C_6 and C_7 remain to be determined.

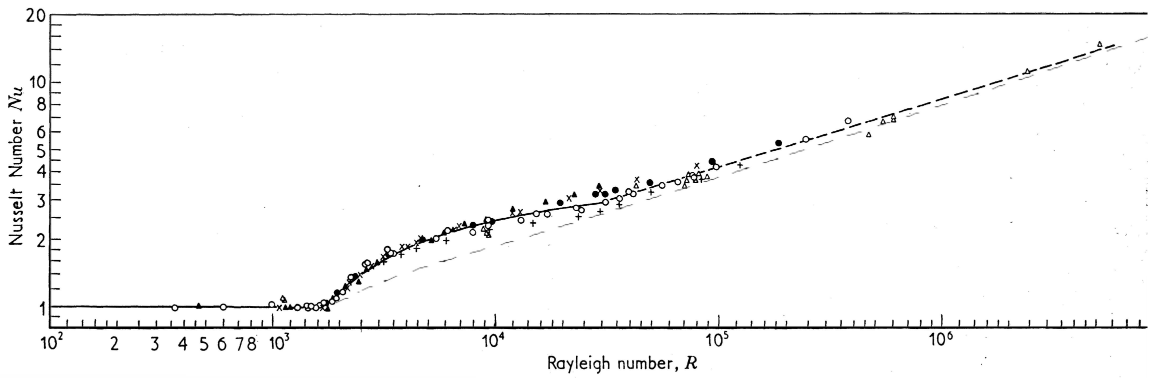

FIGURE 1. The Rayleigh-Nusselt relation adapted from Chandrasekhar (1961). The dashed line is the model prediction for $\Omega = 0, \nu = 10^{-3}, \kappa = 10^{-4}$ generated by varying the temperature difference between the two plates.

the boundaries, implying $R_{ij} = F_i = 0$. At the lower boundary $\Theta = \Delta T$ and at the upper boundary $\Theta = 0$. Since the temperature perturbations are zero at both boundaries, $Q = 0$.

In Figure 1 we show the Nusselt number - Rayleigh number relationship in the non-rotating case for both our model and the selection of experimental data summarised by Chandrasekhar (1961). Note that our model shows good agreement with the experimental data for the critical Rayleigh number where the transition between conductive (Nu = 1) and convective (Nu > 1) heat transport occurs. Our model also reproduces the standard power law relationship $\mathrm{Nu} \propto \mathrm{Ra}^{1/3}$ at high Rayleigh number which is a natural consequence of our selection $L = d$ when $\Omega = 0$ (cf. Prandtl's mixing length theory). However, the good quantitative agreement between the experimental data and the model prediction for Nu was unexpected since C_6 or C_7, which play a role when the system is convective, have not yet been adjusted from their default value of unity. By adjusting these we should be able to further improve the correspondence of the model to reality for $\mathrm{Ra} > \mathrm{Ra_c}$.

In rotating systems, convection is known to delay of the onset of convection at high Taylor numbers. Linear theory (cf. Chandrasekhar (1961)) predicts that $\mathrm{Ra_c} \propto \mathrm{Ta}^{2/3}$, with a coefficient of proportionality which depends somewhat on the boundary conditions and on the Prandtl number of the system. In Figure 2, we compare this known relationship to our model (solid line). The predicted power law matches linear theory, but the coefficient of proportionality is somewhat smaller than required. Nonetheless, we consider the agreement satisfactory considering the simplicity of this closure model.

To conclude, we find that the closure model adequately describes both the turbulent convective heat flux as a function of Rayleigh number in the absence of rotation as well as the delay of the onset of convection in a rotating system. In future work, we intend to compare the degree to which the model matches numerical simulations of developed convection in the rotating Rayleigh-Benard problem (e.g. Julien et al. (1996)). This comparison will be useful in determining the utility of our model away from the onset of convection in a rotating system and permit the calibration of C_6 and C_7, the remaining free parameters of the model.

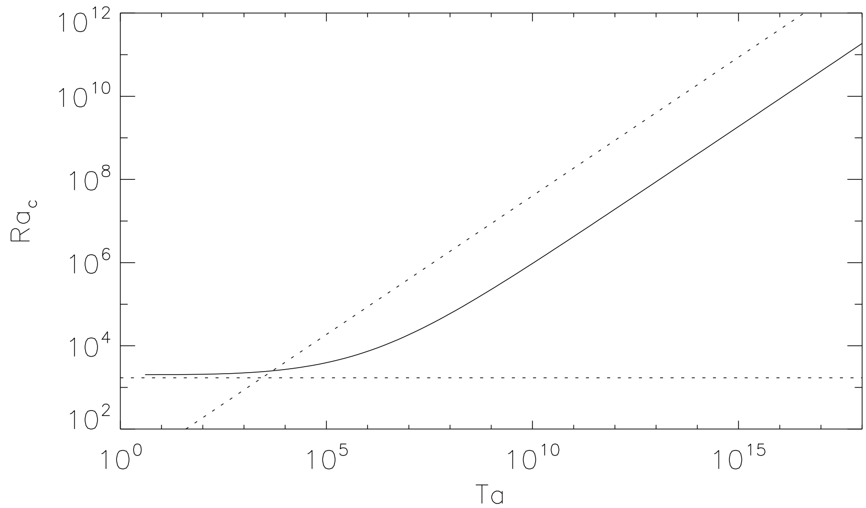

FIGURE 2. The Taylor-Rayleigh critical relation. The solid line is the model prediction for the critical Rayleigh number as a function of Taylor number with $\nu = 10^{-3}, \kappa = 10^{-4}$. The dotted lines are the asymptotic relations derived from linear stability analysis by Chandrasekhar (1961) for $\Omega = 0$ and for Ta $\rightarrow \infty$ for the direct mode of instability (see equation 184 on page 106).

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