Discontinuous Dynamical Systems

A tutorial on notions of solutions, nonsmooth analysis, and stability

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This article considers discontinuous dynamical systems. By discontinuous we mean that the vector field defining the dynamical system can be a discontinuous function of the state. Specifically, we consider dynamical systems of the form

$$\dot{x}(t) = X(t, x(t)),$$

with $X : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, $d \in \mathbb{N}$. For each fixed $t \in \mathbb{R}$, the function $x \mapsto X(t, x)$ is not necessarily continuous.

Discontinuous dynamical systems arise in a large number of applications. In optimal control problems, open-loop bang-bang controllers switch discontinuously between extremum values of the bounded inputs to generate minimum-time trajectories from one state to another. Thermostats implement closed-loop bang-bang controllers to regulate room temperature. In nonsmooth mechanics, the evolution of rigid bodies is subject to velocity jumps and force discontinuities as a result of friction and impacts. The robotic manipulation of objects with mechanical contacts or the motion of vehicles in land, aerial and underwater terrains are yet two more examples where discontinuities naturally occur from the interaction with the environment.

Other times discontinuities are engineered by design. This is the case, for instance, in the stabilization of control systems. The theory of sliding mode control has developed a systematic approach to the design of discontinuous feedback controllers for stabilization. A result due to Brockett [1, 2, 3] implies that many control systems, including driftless systems, cannot be stabilized by means of continuous state-dependent feedbacks. As a result, one is forced to consider either time-dependent or discontinuous feedbacks. The application of Milnor's theorem to

the characterization of the domain of attraction of an asymptotically stable vector field (see [2]) implies that, in environments with obstacles, globally stabilizing controllers must be necessarily discontinuous. In behavior-based robotics, researchers seek to induce global emerging behaviors in the overall network by prescribing interaction rules among individual robots. Simple laws such as "move away from the closest other robot or environmental boundary" give rise to discontinuous dynamical systems. In optimization problems, the design of gradient-like continuous-time algorithms to optimize nonsmooth objective functions often gives rise to discontinuous dynamical systems. The range of applications where discontinuous systems have been employed goes beyond control, robotics and mechanics, and includes examples from linear algebra, queuing theory, cooperative control and a large etcetera. The interested reader can find in the literature more exotic examples.

Independently of the particular application in mind, one always faces similar questions when dealing with discontinuous dynamical systems. The most basic one is the notion of solution. Since the vector field is discontinuous, continuously differentiable curves that satisfy the associated dynamical system do not exist in general, and we must face the issue of identifying a suitable notion of solution. A look into the literature reveals that there is not a unique answer to this question. Depending on the specific problem at hand, authors have used different notions. Let us comment on this in more detail.

Caratheodory solutions are the most natural generalization of the classical notion of solution. Roughly speaking, one proceeds by allowing classical solutions not to follow the direction of the vector field at a few time instants. However, Caratheodory solutions do not exist in many of the applications detailed above. The reason is that their focus on the value of the vector field at each specific point makes them too rigid to cope with the discontinuities.

Filippov and Krasovskii solutions, instead, make use of the concept of differential inclusion. To define a differential inclusion, one makes use of set-valued maps. Just as a (standard) map takes a point in some space to *a* point in some other space, a *set-valued map* takes a point in some space to *a set of points* in some other space. Note that a (standard) map can be seen as a set-valued map that takes points to singletons, that is, sets comprised of a single point. A differential inclusion is then an equation that specifies that the state derivative must belong to a set of directions, rather than be the specific direction determined by the vector field. This flexibility is key in providing general conditions on the vector field under which Filippov and Krasovskii solutions exist. These solution notions play a key role in many of the applications mentioned above, including mechanics with friction and sliding mode control.

However, the Brockett's impossibility theorem also holds when solutions are understood in either the Filippov or the Krasovskii sense. The notion of sample-and-hold solution turns out to be the appropriate one to circumvallate Brockett's theorem, and establish the equivalence between asymptotic controllability and feedback stabilization. Euler solutions are –very much like in the

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case in which X is continuous– useful in establishing existence results, and in characterizing basic mathematical properties of the dynamical system.

Other notions that can be found in the literature include the ones proposed by Hermes [4, 5], Ambrosio [6], Sentis [7], and Yakubovich-Leonov-Gelig [8] solutions, see Table 1. The Russian literature is full of different notions of solutions for discontinuous systems, see [8, Section 1.1.3]. The notions of Caratheodory, Euler, sample-and-hold, Filippov and Krasovskii solutions are compared in [9]. The notions of Hermes, Filippov and Krasovskii solutions are compared in [5]. The notions of Caratheodory, Euler and Sentis solutions are compared in [10]. For reasons of space and relevance, we have chosen to focus here on Caratheodory, Filippov and sample-and-hold solutions. Most of the discussion regarding Filippov solutions can be easily transcribed to Krasovskii solutions.

Notion of solution	References
Caratheodory	[11]
Filippov	[11]
Krasovskii	[12]
Euler	[11, 13]
Sample-and-hold	[14]
Hermes	[4, 5]
Sentis	[7, 10]
Ambrosio	[6]
Yakubovich-Leonov-Gelig	[8]

Table 1. Several notions of solution for discontinuous dynamics. Depending on the specific problem, some notions give more physically meaningful solution trajectories than others.

Once the notion of solution has been settled, there are a number of natural questions that follow, including uniqueness, continuous dependence with respect to initial conditions, stability and asymptotic convergence. Here, we pay special attention to the uniqueness of solutions and to the stability analysis. For ordinary differential equations, it is well-known that the continuity of the vector field does not guarantee uniqueness of solutions. Likewise, for discontinuous vector fields, uniqueness of solutions is not guaranteed in general either –no matter what notion of solution is chosen. Along the discussion, we report a number of sufficient conditions for uniqueness. We also present results specifically tailored to piecewise continuous vector fields and differential inclusions.

The lack of uniqueness of solutions generally requires a little bit of extra analysis because, if we try to establish a specific property of a dynamical system, we need to take into account the possibly multiple solutions starting from each initial condition. This multiplicity leads us to consider standard concepts like invariance or stability together with the adjectives weak and strong. Roughly speaking, "weak" is used when the specific property is satisfied by at least one solution starting from each initial condition. On the other hand, "strong" is used when the specific property is satisfied by all solutions starting from each initial condition.

We present weak and strong stability results for discontinuous dynamical systems and differential inclusions. As we justify later in the example of the nonsmooth harmonic oscillator, the family of smooth Lyapunov functions is not rich enough to handle the stability analysis of discontinuous systems. This fact leads naturally to the study of tools from nonsmooth analysis. In particular, we pay special attention to the concepts of generalized gradient of locally Lipschitz functions and proximal subdifferential of lower semicontinuous functions. Building on these notions, one can establish weak and strong monotonic properties of candidate Lyapunov functions along the solutions of discontinuous dynamical systems. These results are later key in providing suitable generalizations of Lyapunov stability theorems and the LaSalle Invariance Principle. We illustrate the applicability of these results in a class of nonsmooth gradient flows.

There are two ways of invoking the stability results presented here when dealing with control systems: (i) by choosing a specific input function and considering the resulting dynamical system, or (ii) by associating to the control system the set-valued map that maps each state to the set of all vectors generated by the allowable inputs, and considering the resulting differential inclusion. The latter viewpoint allows to consider the full range of possibilities of the control system viewed as a whole, since it does not necessarily focus on a particular choice of controls. This approach also allows us to use nonsmooth stability tools developed for differential inclusions in the analysis of control systems. We explore this idea in detail later in the article.

The topics treated here could be explored in more detail. Given the large body of work on discontinuous systems and the limited space of the article, we have tried to provide a clear exposition of a few useful and important results. Additionally, there are various relevant issues that are left out in the exposition. An incomplete list includes the study of the continuous dependence of solutions with respect to initial conditions, the characterization of robustness properties, converse Lyapunov theorems, and measure differential inclusions. The interested reader may consult [3, 11, 13, 15, 16, 17] and references therein to further explore these topics. Also, we do not cover any viability theory, see [18], discuss notions of solution for systems that involve both continuous and discrete time, see, for instance [19, 20, 21], or consider numerical methods for discontinuous systems and differential inclusions, see for example [22, 23, 24].

The article is organized as follows. We start by reviewing the basic results on existence and uniqueness of (continuously differentiable) solutions of ordinary differential equations. We also provide several examples of the different situations that arise when the vector field fails to satisfy the required smoothness properties. Next, we introduce three representative examples of discontinuous systems: the brick on a frictional ramp, the nonsmooth harmonic oscillator and the "move-away-from-closest-neighbor" interaction law. We then introduce various notions of solution for discontinuous systems, discuss existence and uniqueness results, and present various useful tools for analysis. In preparation for the statement of stability results, we introduce the generalized gradient and the proximal subdifferential notions from nonsmooth analysis, and present various tools for their explicit computation. Then, we develop analysis results to characterize the stability and asymptotic convergence properties of the solutions of discontinuous dynamical systems. We

illustrate these nonsmooth stability results in various examples, paying special attention to gradient systems. We end the article with some concluding remarks. Throughout the discussion, we interchangeably use "differential equation," "dynamical system," and "vector field."

Two final remarks regarding non-autonomous differential equations and the domain of definition of the vector fields. Most of the exposition contained here can be carried over to the time-dependent setting, generally by treating time as a parameter. To simplify the presentation, we have chosen to discuss non-autonomous vector fields only when introducing the various notions of solution for discontinuous systems. The rest of the presentation focuses on autonomous vector fields. Likewise, for simplicity, we have chosen to consider vector fields defined over the whole Euclidean space, although most of the exposition here can be carried out in a more general setup.

Ordinary Differential Equations: Existence and Uniqueness of Solutions, and Some Counterexamples

In this section, we review some of the basic results on existence and uniqueness of solutions for ordinary differential equations (ODEs). We also present examples that do not verify the hypotheses of these results but still enjoy existence and uniqueness of solutions, as well as other examples that do not enjoy such desirable properties.

Existence of solutions

Let $X : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be a (non-autonomous) vector field, and consider the differential equation on $\mathbb{R}^d$

$$\dot{x}(t) = X(t, x(t)). \tag{1}$$

A point $x_* \in \mathbb{R}^d$ is an *equilibrium* of the differential equation if $0 = X(t, x_*)$ for all $t \in \mathbb{R}$. A *solution of* (1) on $[t_0, t_1]$ is a continuously differentiable map $\gamma : [t_0, t_1] \to \mathbb{R}^d$ such that $\dot{\gamma}(t) = X(t, \gamma(t))$. Usually, we refer to γ as a solution with initial condition $\gamma(t_0) = x_0$. If the vector field is autonomous, that is, does not depend explicitly on time, then without loss of generality we take $t_0 = 0$. A solution is *maximal* if it cannot be extended, that is, if it is not the result of the truncation of another solution with a larger interval of definition. Note that the interval of definition of a maximal solution might be right half-open.

Essentially, continuity of the vector field suffices to guarantee the existence of solutions, as the following result states.

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Proposition 1. *Let $X : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. Assume that (i) for each $t \in \mathbb{R}$, the map $x \mapsto X(t, x)$ is continuous, (ii) for each $x \in \mathbb{R}^d$, the map $t \mapsto X(t, x)$ is measurable, and (iii) X is locally bounded, that is, for all $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, there exist $\varepsilon \in (0, \infty)$ and an integrable function $m : [t, t + \delta] \to (0, \infty)$ such that $\|X(s, y)\|_2 \le m(s)$ for all $s \in [t, t + \delta]$ and all $y \in B(x, \varepsilon)$. Then, for any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$, there exists a solution of (1) with initial condition $x(t_0) = x_0$.*

For autonomous vector fields, Proposition 1 takes a simpler form: $X : \mathbb{R}^d \to \mathbb{R}^d$ must simply be continuous in order to have at least a solution starting from any given initial condition. As the following example shows, if the vector field is discontinuous, then solutions of (1) might not exist.

Discontinuous vector field with non-existence of solutions

Consider the autonomous vector field $X : \mathbb{R} \to \mathbb{R}$,

$$X(x) = \begin{cases} -1, & x > 0, \\ 1, & x \le 0. \end{cases} \tag{2}$$

This vector field is discontinuous at 0 (see Figure 1(a)). The associated dynamical system $\dot{x}(t) = X(x(t))$ does not have a solution starting from 0. That is, there does not exist a continuously differentiable map $\gamma : [0, t_1] \to \mathbb{R}$ such that $\dot{\gamma}(t) = X(\gamma(t))$ and $\gamma(0) = 0$. Otherwise, if such a solution exists, then $\dot{\gamma}(0) = 1$, and $\dot{\gamma}(t) = -1$ for any positive t sufficiently small, which contradicts the fact that $\dot{\gamma}$ is continuous.

However, the following example shows that the lack of continuity of the vector field does not necessarily preclude the existence of solutions.

Discontinuous vector field with existence of solutions

Consider the autonomous vector field $X : \mathbb{R} \to \mathbb{R}$,

$$X(x) = -\operatorname{sign}(x) = \begin{cases} -1, & x > 0, \\ 0, & x = 0, \\ 1, & x < 0. \end{cases} \tag{3}$$
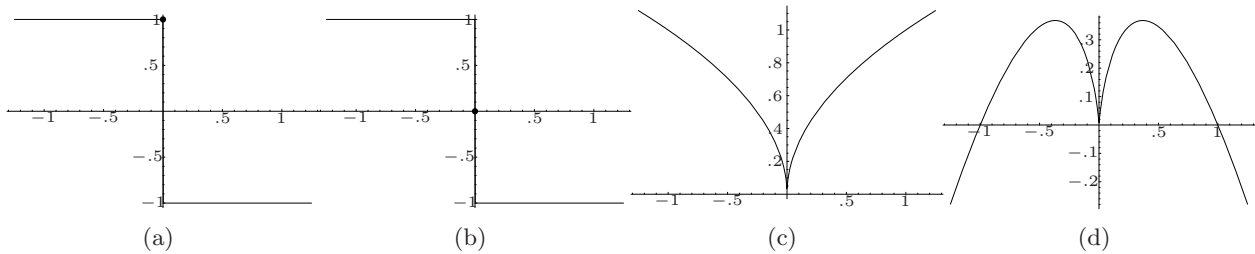
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Figure 1. Discontinuous −(a) and (b)− and not-locally Lipschitz −(c) and (d)− vector fields. The vector fields in (a) and (b) do not verify the hypotheses of Proposition 1, and therefore the existence of solutions is not guaranteed. The vector field in (a) has no solution starting from 0**. However, the vector field in (b) has a solution starting from any initial condition. The vector fields in (c) and (d) do not verify the hypotheses of Proposition 2, and therefore the uniqueness of solutions is not guaranteed. The vector field in (c) has two solutions starting from** 0**. However, the vector field in (d) has a unique solution starting from any initial condition.**

This vector field is discontinuous at 0 (see Figure 1(b)). However, the associated dynamical system $\dot{x}(t) = X(x(t))$ has a solution starting from each initial condition. Specifically, the maximal solutions are

$$
\begin{array}{lll}
\text{For } x(0) > 0, & \gamma : [0, x(0)) \to \mathbb{R}, & \gamma(t) = x(0) - t, \\
\text{For } x(0) = 0, & \gamma : [0, \infty) \to \mathbb{R}, & \gamma(t) = 0, \\
\text{For } x(0) < 0, & \gamma : [0, -x(0)) \to \mathbb{R}, & \gamma(t) = x(0) + t.
\end{array}
$$

The difference between the vector fields (2) and (3) is minimal (they are equal up to the value at 0), and yet the question of the existence of solutions has a different answer for each of them. We see later how considering a different notion of solution can reconcile the answers given to the existence question for these vector fields.

Uniqueness of solutions

Next, let us turn our attention to the issue of uniqueness of solutions. Here and in what follows, *(right) uniqueness* means that, if there exist two solutions with the same initial condition, then they coincide on the intersection of their intervals of existence. Formally, if $\gamma_1 : [t_0, t_1] \to \mathbb{R}^d$ and $\gamma_2 : [t_0, t_2] \to \mathbb{R}^d$ are solutions of (1) with $\gamma_1(t_0) = \gamma_2(t_0)$, then uniqueness means that $\gamma_1(t) = \gamma_2(t)$ for all $t \in [t_0, t_1] \cap [t_0, t_2] = [t_0, \min\{t_1, t_2\}]$. The following result provides a sufficient condition for uniqueness.

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Proposition 2. *Under the hypotheses of Proposition 1, further assume that for all $x \in \mathbb{R}^d$, there exist $\varepsilon \in (0, \infty)$ and an integrable function $L_x : \mathbb{R} \to (0, \infty)$ such that*

$$(X(t, y) - X(t, y'))^T (y - y') \le L_x(t) \, \|y - y'\|_2^2, \tag{4}$$

for all $y, y' \in B(x, \varepsilon)$ and all $t \in \mathbb{R}$. Then, for any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$, there exists a unique solution of (1) with initial condition $x(t_0) = x_0$.

Equation (4) is usually referred to as a *one-sided Lipschitz condition*. In particular, it is not difficult to see that locally Lipschitz vector fields (see the sidebar "Locally Lipschitz Functions") verify this condition. The opposite is not true (as an example, consider the vector field $X : \mathbb{R} \to \mathbb{R}$ defined by $X(x) = x \log(|x|)$ for $x \ne 0$ and $X(0) = 0$, which verifies the one-sided Lipschitz condition (4) around 0, but is not locally Lipschitz at 0). Locally Lipschitzness is the most common requirement invoked to guarantee uniqueness of solution. As Proposition 2 shows, uniqueness is indeed guaranteed under slightly more general conditions.

The following example shows that, if the hypotheses of Proposition 2 are not verified, then solutions might not be unique.

Continuous, not locally Lipschitz vector field with non-uniqueness of solutions

Consider the autonomous vector field $X : \mathbb{R} \to \mathbb{R}$,

$$X(x) = \sqrt{|x|}. \tag{5}$$

This vector field is continuous everywhere, and locally Lipschitz on $\mathbb{R} \setminus \{0\}$ (see Figure 1(c)). Even more, X does not verify equation (4) in any neighborhood of 0. The associated dynamical system $\dot{x}(t) = X(x(t))$ has two maximal solutions starting from 0, namely,

$$\gamma_1 : [0, \infty) \to \mathbb{R}, \qquad \gamma_1(t) = 0,$$
$$\gamma_2 : [0, \infty) \to \mathbb{R}, \qquad \gamma_2(t) = t^2/4.$$

However, there are cases where the hypotheses of Proposition 2 are not verified, and the differential equation still enjoys uniqueness of solution, as the following example shows.

Continuous, not locally Lipschitz vector field with uniqueness of solutions

Consider the autonomous vector field $X : \mathbb{R} \to \mathbb{R}$,

$$X(x) = \begin{cases} -x \log x, & x > 0, \\ 0, & x = 0, \\ x \log(-x), & x < 0. \end{cases} \tag{6}$$

This vector field is continuous everywhere, and locally Lipschitz on $\mathbb{R} \setminus \{0\}$ (see Figure 1(d)). Even more, X does not verify equation (4) in any neighborhood of 0. However, the associated dynamical system $\dot{x}(t) = X(x(t))$ has a unique solution starting from each initial condition. Specifically, the maximal solution is

For $x(0) > 0$, $\gamma : [0, \infty) \to \mathbb{R}$, $\gamma(t) = \exp(\log x(0) \exp(-t))$,
For $x(0) = 0$, $\gamma : [0, \infty) \to \mathbb{R}$, $\gamma(t) = 0$,
For $x(0) < 0$, $\gamma : [0, \infty) \to \mathbb{R}$, $\gamma(t) = -\exp(\log(-x(0)) \exp(t))$.

Note that the statement of Proposition 2 prevents us from applying it to discontinuous vector fields, since solutions are not even guaranteed to exist. However, the discontinuous vector field (3) verifies the one-sided Lipschitz condition around any point, and indeed, the associated dynamical system enjoys uniqueness of solutions. A natural question is then to ask under what conditions discontinuous vector fields have a unique solution starting from each initial condition. Of course, the answer to this question relies on the notion of solution itself. We explore in detail these questions in the section entitled "Notions of Solution for Discontinuous Dynamical Systems."

Examples of Discontinuous Dynamical Systems

In this section we present three more examples of discontinuous dynamical systems. These examples, together with the ones discussed in above, motivate the extension of the classical notion of (continuously differentiable) solution for ordinary differential equations, which is the subject of the next section.

Brick on a frictional ramp

Consider a brick sliding on a ramp, an example taken from [16]. As the brick moves down, it experiments a friction force in the opposite direction as a result of the contact with the ramp (see Figure 2(a)). Coulomb's friction law is the most accepted model of friction available.
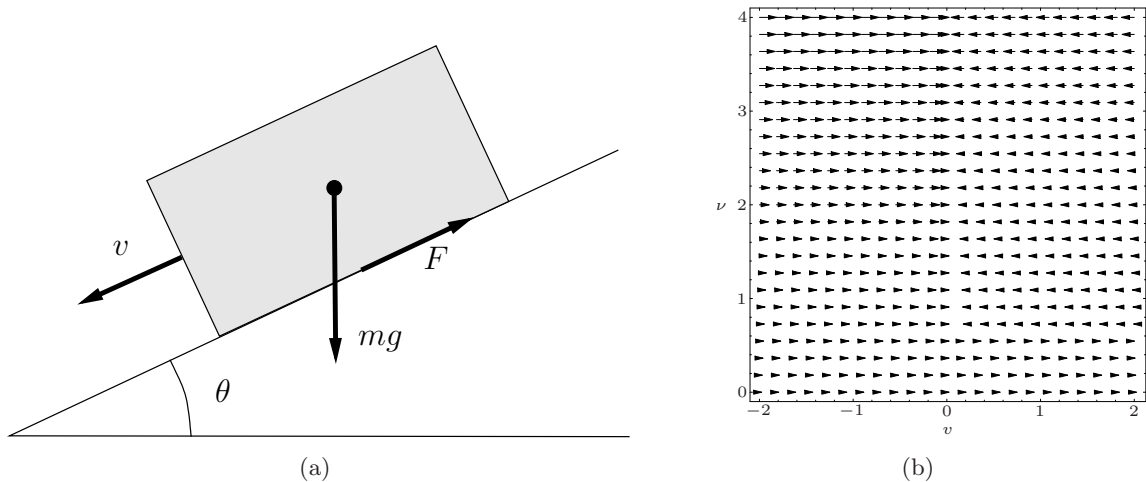
Figure 2. Brick sliding on a frictional ramp. The plot in (a) shows the physical quantities used to describe the example. The plot in (b) shows the one-dimensional phase portraits of (7) **corresponding to values of the friction coefficient between** 0 **and** 4**, with a constant ramp incline of** $\pi/6$**.**

In its simplest form, it says that the friction force is bounded in magnitude by the normal contact force times the coefficient of friction.

The application of Coulomb's law to the brick example gives rise to the equation of motion

$$m\frac{dv}{dt} = mg\sin\theta - \nu mg\cos\theta\,\mathrm{sign}(v), \tag{7}$$

where m and v are the mass and velocity of the brick, respectively, g is the constant of gravity, θ is the incline of the ramp, and ν is the coefficient of friction. The right-hand side of this equation is clearly a discontinuous function of v. Figure 2(b) shows the phase plot of this system for different values of the friction coefficient.

Depending on the magnitude of the friction force, one may observe in real experiments that the brick stops and stays stopped. In other words, the brick attains $v = 0$ in finite time, and stays with $v = 0$ for certain period of time. The classical solutions of this differential equation do not exhibit this type of behavior. To see this, note the similarity of (7) and (3). In order to explain this type of physical evolutions, we need then to understand the discontinuity of the equation, and expand our notion of solution beyond the classical one.

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Nonsmooth harmonic oscillator

Arguably, the harmonic oscillator is one of the most encountered examples in textbooks of periodic behavior in physical systems. Here, we introduce a nonsmooth version of it, following [25]. Consider a mechanical system with two degrees of freedom, evolving according to

$$\dot{x}_1(t) = \text{sign}(x_2(t)),$$
$$\dot{x}_2(t) = -\text{sign}(x_1(t)).$$

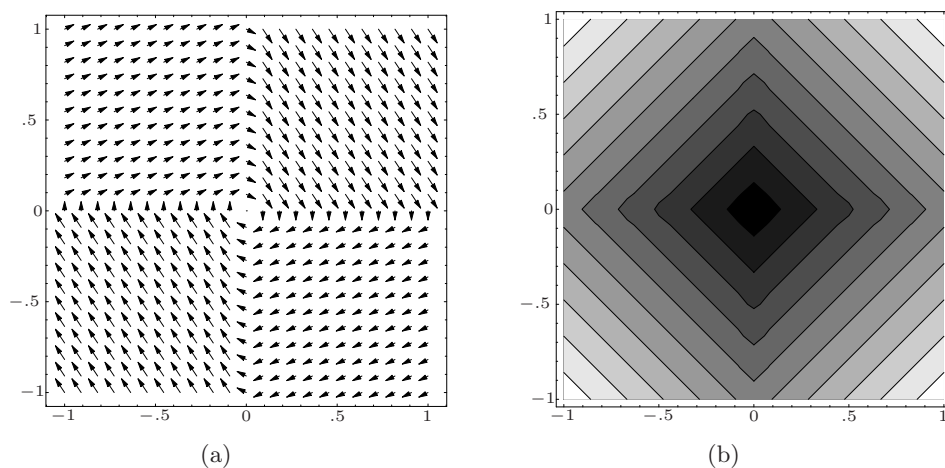The phase portrait of this system is plotted in Figure 3(a).

(a)　　　　　　　　　　　(b)

Figure 3. Nonsmooth harmonic oscillator. The plot in (a) shows the phase portrait on $[-1,1]^2$ of the vector field $(x_1, x_2) \mapsto (\text{sign}(x_2), -\text{sign}(x_1))$, and the plot in (b) shows the contour plot on $[-1,1]^2$ of the function $(x_1, x_2) \mapsto |x_1| + |x_2|$.

By looking at the equations of motion, $(0,0)$ is the unique equilibrium point of the system. Regarding other initial conditions, it seems clear how the system evolves while not in any of the coordinate axes. However, things are not so clear on the axes. If we perform a discretization of the equations of motion, and make the time stepsize smaller and smaller, we find that the trajectories look closer and closer to the set of diamonds plotted in Figure 3. These diamonds correspond to the level sets of the function $(x_1, x_2) \mapsto |x_1| + |x_2|$. This observation is analogous to the fact that the level sets of the function $(x_1, x_2) \mapsto x_1^2 + x_2^2$ correspond to the trajectories of the classical harmonic oscillator. However, the diamond trajectories are clearly not continuously differentiable, so to consider them as valid solutions we need a different notion of solution than the classical one.

"Move-away-from-closest-neighbor" interaction law

Consider n nodes $p_1, \ldots, p_n$ evolving in a convex polygon Q according to the interaction rule "move-away-from-closest-neighbor." Formally, let $\mathcal{S} = \{(p_1, \ldots, p_n) \in Q^n \mid p_i = p_j \text{ for some } i \neq j\}$, and consider the *nearest-neighbor map* $\mathcal{N} : Q^n \setminus \mathcal{S} \to Q^n$ defined by

$$\mathcal{N}_i(p_1, \ldots, p_n) \in \operatorname{argmin}\{\|p_i - q\|_2 \mid q \in \partial Q \cup \{p_1, \ldots, p_n\} \setminus \{p_i\}\},$$

where ∂Q denotes the boundary of Q. Note that $\mathcal{N}_i(p_1, \ldots, p_n)$ is one of the closest nodes to p_i, and that the same point can be the closest neighbor to more than one node. Now, consider the "move-away-from-closest-neighbor" interaction law defined by

$$\dot{p}_i = \frac{p_i - \mathcal{N}_i(p_1, \ldots, p_n)}{\|p_i - \mathcal{N}_i(p_1, \ldots, p_n)\|_2}, \quad i \in \{1, \ldots, n\}. \tag{8}$$

Clearly, changes in the nearest-neighbor map induce discontinuities in the dynamical system. Figure 4 shows two instances where these discontinuities occur.
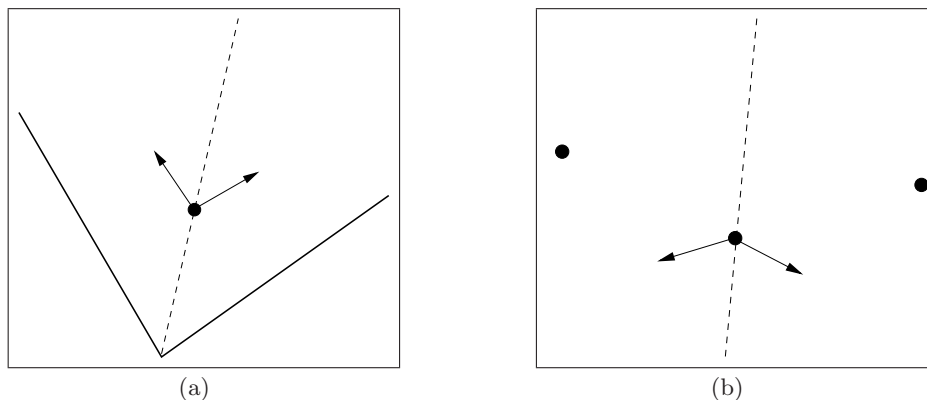

(a)   (b)

Figure 4. "Move-away-from-closest-neighbor" interaction law. The plots in (a) and (b) show two examples of how infinitesimal changes in a node location give rise to different closest neighbors (either polygonal boundaries or other nodes) and hence completely different directions of motion.

To analyze this dynamical system, we need to understand how the discontinuities affect its evolution. It seems reasonable to postulate that the set $Q^n \setminus \mathcal{S}$ remains invariant under this flow, that is, that nodes never run into each other, but we need to extend our notion of solution –and redefine our notion of invariance accordingly– in order to ensure it.

Notions of Solution for Discontinuous Dynamical Systems

In the previous sections, we have seen that the usual notion of solution for ordinary differential equations is too restrictive when considering discontinuous vector fields. Here, we explore other notions of solution to reconcile the mismatch. In general, one may think that a good way of taking care of the discontinuities of the differential equation (1) is by allowing solutions to violate it (that is, do not follow the direction specified by X) at a few time instants. The precise mathematical notion corresponding to this idea is that of Caratheodory solution, which we introduce next.

Caratheodory solutions

A *Caratheodory solution of* (1) defined on $[t_0, t_1] \subset \mathbb{R}$ is an absolutely continuous map $\gamma : [t_0, t_1] \to \mathbb{R}^d$ such that $\dot{\gamma}(t) = X(t, \gamma(t))$ for almost every $t \in [t_0, t_1]$. The sidebar "Absolutely continuous functions" reviews the notion of absolutely continuous function, and examines some of their properties. Arguably, this notion of solution is the most natural candidate for a discontinuous system (indeed, Caratheodory solutions are also called *classical* solutions).

Consider, for instance, the vector field $X : \mathbb{R} \to \mathbb{R}$ defined by

$$X(x) = \begin{cases} 1, & x > 0, \\ \frac{1}{2}, & x = 0, \\ -1, & x < 0. \end{cases}$$

This vector field is discontinuous at 0. The associated dynamical system $\dot{x}(t) = X(x(t))$ does not have a (continuously differentiable) solution starting from 0. However, it has two Caratheodory solutions starting from 0, namely, $\gamma_1 : [0, \infty) \to \mathbb{R}$, $\gamma_1(t) = t$, and $\gamma_2 : [0, \infty) \to \mathbb{R}$, $\gamma_2(t) = -t$. Note that both γ_1 and γ_2 violate the differential equation only at $t = 0$, that is, $\dot{\gamma}_i(0) \neq X(\gamma_i(0))$, for $i = 1, 2$.

However, the good news are over soon. The physical motions observed in the brick sliding on a frictional ramp example, where the brick slides for a while and then stays stopped, are not Caratheodory solutions. The discontinuous vector field (2) does not admit any Caratheodory solution starting from 0. The "move-away-from-closest-neighbor" interaction law is yet another example where Caratheodory solutions do not exist either, as we show next.

For the "move-away-from-closest-neighbor" interaction law, consider one agent moving in the square environment $[-1, 1]^2 \subset \mathbb{R}^2$. Since no other agent is present in the square, the agent just moves away from the closest polygonal boundary, according to the vector field

$$X(x_1, x_2) = \begin{cases} (-1, 0), & -x_1 < x_2 \leq x_1, \\ (0, 1), & x_2 < x_1 \leq -x_2, \\ (1, 0), & x_1 \leq x_2 < -x_1, \\ (0, -1), & -x_2 \leq x_1 < x_2. \end{cases} \tag{9}$$

Since on the diagonals of the square, $\{(a, a) \in [-1, 1]^2 \mid a \in [-1, 1]\} \cup \{(a, -a) \in [-1, 1]^2 \mid a \in [-1, 1]\}$, the "move-away-from-closest-neighbor" interaction law takes multiple values, we have chosen one of them in the definition of X. Figure 5 shows the phase portrait.   The vector field
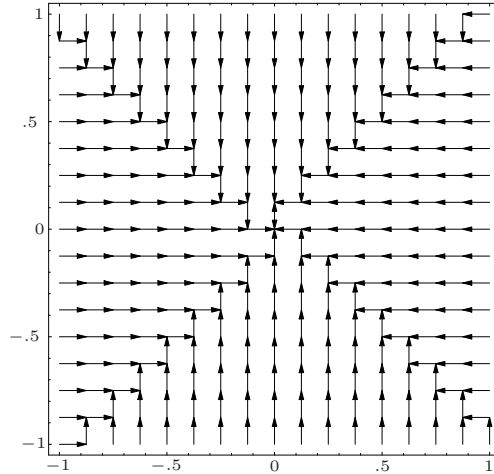

Figure 5. Phase portrait of the "move-away-from-closest-neighbor" interaction law for one agent moving in the square $[-1, 1]^2 \subset \mathbb{R}^2$. Note that there is no Caratheodory solution starting from an initial condition in the diagonals of the square.

X is discontinuous on the diagonals. It is precisely when the initial condition belongs to these diagonals that the dynamical system $\dot{x}(t) = X(x(t))$ does not admit any Caratheodory solution.

Sufficient conditions for the existence of Caratheodory solutions

Specific conditions under which Caratheodory solutions exist have been carefully identified in the literature, see for instance [11], and are known as *Caratheodory conditions*. Actually, they turn out to be a slight generalization of the conditions stated in Proposition 1, as the following result shows.

Proposition 3. *Let $X : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. Assume that (i) for almost all $t \in \mathbb{R}$, the map $x \mapsto X(t,x)$ is continuous, (ii) for each $x \in \mathbb{R}^d$, the map $t \mapsto X(t,x)$ is measurable, and (iii) X is locally essentially bounded, that is, for all $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, there exist $\varepsilon \in (0,\infty)$ and an integrable function $m : [t, t+\delta] \to (0,\infty)$ such that $\|X(s,y)\|_2 \le m(s)$ for all $s \in [t, t+\delta]$ and almost all $y \in B(x, \varepsilon)$. Then, for any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$, there exists a Caratheodory solution of (1) with initial condition $x(t_0) = x_0$.*

Note that in the autonomous case, the assumptions of this result amount to ask the vector field to be continuous. This requirement is no improvement with respect to Proposition 1, since we already know that in the continuous case, continuously differentiable solutions exist. Because of this reason, various authors have explored conditions for the existence of Caratheodory solutions specifically tailored to autonomous vector fields. For reasons of space, we do not go into details here. The interested reader may consult [26] to find that directional continuous vector fields admit Caratheodory solutions, and [27] to learn about patchy vector fields, a special family of autonomous, discontinuous vector fields that also admit Caratheodory solutions.

Caratheodory solutions can also be defined for differential inclusions, instead of differential equations. The sidebars "Set-valued Maps" and "Differential Inclusions and Caratheodory Solutions" explain how in detail.

Given the limitations of the notion of Caratheodory solution, an important research thrust in the theory of differential equations has been the identification of more flexible notions of solution for discontinuous vector fields. Let us discuss various alternatives, and illustrate their advantages and disadvantages.

Filippov solutions

As we have seen when considering the existence of Caratheodory solutions starting from a desired initial condition, focusing on the specific value of the vector field at the initial condition might be too shortsighted. Due to the discontinuities of the vector field, things can be very different arbitrarily close to the initial condition, and this mismatch might indeed make impossible to construct a solution. The vector field in (2) and the "move-away-from-closest-neighbor" interaction law are instances of this situation.

What if, instead of focusing on the value of the vector field at each point, we somehow consider how the vector field looks like *around* each point? The idea of looking at a neighborhood of each point is at the core of the notion of Filippov solution [11]. A closely related notion is that

of Krasovskii solution (to ease the exposition, we do not deal with the latter here. The interested reader is referred to [9, 12]).

The mathematical treatment to formalize this "neighborhood" idea uses set-valued maps. Let us discuss it informally for an autonomous vector field $X : \mathbb{R}^d \to \mathbb{R}^d$. Filippov's idea is to associate a set-valued map to X by looking at the neighboring values of X around each point. Specifically, for $x \in \mathbb{R}^d$, one evaluates the vector field X on the points belonging to $B(x, \delta)$, the open ball centered at x of radius $\delta > 0$. We examine the result when δ gets closer to 0 by performing this operation for increasingly smaller δ. For further flexibility, we may exclude any set of measure zero in the ball $B(x, \delta)$ when evaluating X, so that the outcome is the same for two vector fields that only differ by a set of measure zero.

Mathematically, the previous paragraph can be summarized as follows. For $X : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, define the *Filippov set-valued map* $F[X] : \mathbb{R} \times \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ by

$$F[X](t, x) = \bigcap_{\delta > 0} \bigcap_{\mu(S) = 0} \mathrm{co}\{X(t, B(x, \delta) \setminus S)\}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d.$$

In this formula, co denotes convex closure, and μ denotes the Lebesgue measure. Because of the way the Filippov set-valued map is defined, its value at a point is actually independent of the value of the vector field at that specific point. Note that this definition also works for maps of the form $X : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^m$, where d and m are not necessarily equal.

Let us compute this set-valued map for the vector fields (2) and (3). First of all, note that since both vector fields only differ at 0 (that is, at a set of measure zero), their associated Filippov set-valued maps are identical. Specifically, $F[X] : \mathbb{R} \to \mathfrak{B}(\mathbb{R})$ with

$$F[X](x) = \begin{cases} -1, & x > 0, \\ [-1, 1], & x = 0, \\ 1, & x < 0. \end{cases}$$

Now we are ready to handle the discontinuities in the vector field X. We do so substituting the differential equation $\dot{x}(t) = X(t, x(t))$ by the differential inclusion

$$\dot{x}(t) \in F[X](t, x(t)), \tag{10}$$

and considering the solutions of the latter, as defined in the sidebar "Differential Inclusions and Caratheodory Solutions." A *Filippov solution of* (1) defined on $[t_0, t_1] \subset \mathbb{R}$ is a solution of the differential inclusion (10), that is, an absolutely continuous map $\gamma : [t_0, t_1] \to \mathbb{R}^d$ such that $\dot{\gamma}(t) \in F[X](t, \gamma(t))$ for almost every $t \in [t_0, t_1]$, see the sidebar "Differential Inclusions and Caratheodory solutions." Because of the way the Filippov set-valued map is defined, note that any vector field that differs from X in a set of measure zero has the same set-valued map, and hence the same set of solutions. The next result establishes mild conditions under which Filippov solutions exist.

Proposition 4. *For $X : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ measurable and locally essentially bounded, there exists at least a Filippov solution of* (1) *starting from any initial condition.*

The hypotheses of this proposition on the vector field guarantee that the associated Filippov set-valued map verifies all hypothesis of Proposition S1, and hence the existence of solutions follows. As an application of this result, since the autonomous vector fields in (2) and (3) are bounded, Filippov solutions exist starting from any initial condition. Both vector fields have the same (maximal) solutions. Specifically,

$$
\begin{aligned}
&\text{For } x(0) > 0, &&\gamma : [0, \infty) \to \mathbb{R}, &&\gamma(t) = |x(0) - t|_+, \\
&\text{For } x(0) = 0, &&\gamma : [0, \infty) \to \mathbb{R}, &&\gamma(t) = 0, \\
&\text{For } x(0) < 0, &&\gamma : [0, \infty) \to \mathbb{R}, &&\gamma(t) = |x(0) + t|_-.
\end{aligned}
$$

Following a similar line of reasoning, one can show that the physical motions observed in the brick sliding on a frictional ramp example, where the brick slides for a while and then stays stopped, are indeed Filippov solutions. Similar computations can be made for the "move-away-from-closest-neighbor" interaction law to show that Filippov solutions exist starting from any initial condition, as we show next.

"Move-away-from-closest-neighbor" interaction law for one agent in a square –revisited

Consider again the discontinuous vector field for one agent moving in a square under the "move-away-from-closest-neighbor" interaction law. The corresponding set-valued map $F[X] : [-1, 1]^2 \to \mathfrak{B}(\mathbb{R}^2)$ is given by

$$
F[X](x_1, x_2) = \begin{cases}
\{(y_1, y_2) \in \mathbb{R}^2 \mid |y_1 + y_2| \le 1, |y_1 - y_2| \le 1\}, & (x_1, x_2) = (0, 0), \\
\{(-1, 0)\}, & -x_1 < x_2 < x_1, \\
\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 + y_2 = -1, y_1 \in [-1, 0]\}, & 0 < x_2 = x_1, \\
\{(0, 1)\}, & x_2 < x_1 < -x_2, \\
\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 - y_2 = -1, y_1 \in [-1, 0]\}, & 0 < -x_1 = x_2, \\
\{(1, 0)\}, & x_1 < x_2 < -x_1, \\
\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 + y_2 = 1, y_1 \in [0, 1]\}, & x_2 = x_1 < 0, \\
\{(0, -1)\}, & -x_2 < x_1 < x_2, \\
\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 - y_2 = 1, y_1 \in [0, 1]\}, & 0 < x_1 = -x_2.
\end{cases}
$$

According to Proposition 4, since X is bounded, Filippov solutions exist. In particular, the solutions starting from any point in a diagonal are nice straight lines flowing along the diagonal itself and reaching $(0,0)$. For example, the maximal solution $\gamma : [0, \infty) \to \mathbb{R}^2$ starting from $(a, a) \in \mathbb{R}^2$ is

$$t \mapsto \gamma(t) = \begin{cases} (a - \operatorname{sign}(a)t, a - \operatorname{sign}(a)t), & t \leq |a|, \\ (0, 0), & t \geq |a|. \end{cases}$$

Note that the behavior of this solution is quite different from what one might expect by looking at the vector field at the points of continuity. Indeed, the solution *slides* along the diagonals, following a convex combination of the limiting values of X around them, rather the direction specified by X itself. We study in more detail this type of behavior in the section entitled "Piecewise continuous vector fields and sliding motions."

Relationship between Caratheodory and Filippov solutions

One may pose the question: how are Caratheodory and Filippov solutions related? The answer is that not much. An example of a vector field for which both notions of solution exist but Filippov solutions are not Caratheodory solutions is given in [27]. The converse is not true either. For instance, the vector field

$$X(x) = \begin{cases} 1, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

has $t \mapsto 0$ as a Caratheodory solution starting from 0. However, the associated Filippov set-valued map is $F[X] : \mathbb{R} \to \mathfrak{B}(\mathbb{R})$, $F[X](x) = \{1\}$, and hence the unique Filippov solution starting from 0 is $t \mapsto t$. On a related note, Caratheodory solutions are always Krasovskii solutions (but the converse is not true, see [9]).

Computing the Filippov set-valued map

In general, computing the Filippov set-valued map can be a daunting task. The work [28] develops a calculus that simplifies its calculation. We summarize here some useful facts.

Consistency. For $X : \mathbb{R}^d \to \mathbb{R}^d$ continuous at $x \in \mathbb{R}^d$,

$$F[X](x) = \{X(x)\}. \tag{11}$$

Sum rule. For $X_1, X_2 : \mathbb{R}^d \to \mathbb{R}^m$ locally bounded at $x \in \mathbb{R}^d$,

$$F[X_1 + X_2](x) \subset F[X_1](x) + F[X_2](x). \tag{12}$$

Moreover, if one of the vector fields is continuous at x, then equality holds.

Product rule. For $X_1 : \mathbb{R}^d \to \mathbb{R}^m$ and $X_2 : \mathbb{R}^d \to \mathbb{R}^n$ locally bounded at $x \in \mathbb{R}^d$,

$$F[(X_1, X_2)](x) \subset F[X_1](x) \times F[X_2](x). \tag{13}$$

Moreover, if one of the vector fields is continuous at x, then equality holds;

Chain rule. For $Y : \mathbb{R}^d \to \mathbb{R}^n$ continuously differentiable at $x \in \mathbb{R}^d$ with rank n, and $X : \mathbb{R}^n \to \mathbb{R}^m$ locally bounded at $Y(x) \in \mathbb{R}^n$,

$$F[X \circ Y](x) = F[X](Y(x)). \tag{14}$$

Matrix transformation rule. For $X : \mathbb{R}^d \to \mathbb{R}^m$ locally bounded at $x \in \mathbb{R}^d$ and $Z : \mathbb{R}^d \to \mathbb{R}^{d \times m}$ continuous at $x \in \mathbb{R}^d$,

$$F[Z\,X](x) = Z(x)\,F[X](x). \tag{15}$$

Similar expressions can be developed for non-autonomous vector fields.

We conclude this section with an alternative description of the Filippov set-valued map. For $X : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ measurable and locally essentially bounded, one can show that, for each $t \in \mathbb{R}$, there exists $S_t \subset \mathbb{R}^d$ of measure zero such that

$$F[X](t, x) = \mathrm{co}\{\lim_{i \to \infty} X(t, x_i) \mid x_i \to x\,,\ x_i \notin S \cup S_t\},$$

where S is any set of measure zero. As we see later when discussing nonsmooth functions, this description has a remarkable parallelism with the notion of generalized gradient of a locally Lipschitz function.

Piecewise continuous vector fields and sliding motions

When dealing with discontinuous dynamics, one often encounters vector fields that are continuous everywhere except at a surface of the state space. Indeed, the examples of discontinuous

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vector fields that we have introduced so far all fall into this situation. This problem can be naturally interpreted by considering two continuous dynamical systems, each one defined on one side of the surface, glued together to give rise to a discontinuous dynamical system on the overall state space. Here, we analyze the properties of the Filippov solutions in this sort of (quite common) situations.

Let us consider a *piecewise continuous* vector field $X : \mathbb{R}^d \to \mathbb{R}^d$. Here, by piecewise continuous we mean that there exists a finite collection of disjoint *domains* $\mathcal{D}_1, \ldots, \mathcal{D}_m \subset \mathbb{R}^d$ (that is, open and connected sets) that partition $\mathbb{R}^d$ (that is, $\mathbb{R}^d = \cup_{k=1}^m \overline{\mathcal{D}}_k$) such that the vector field X is continuous on each $\overline{\mathcal{D}}_k$, for $k \in \{1, \ldots, m\}$. More general definitions are also possible (by considering, for instance, non-autonomous vector fields), but we restrict our attention to this one for simplicity. Clearly, a point of discontinuity of X must belong to one of the boundaries of the sets $\mathcal{D}_1, \ldots, \mathcal{D}_m$. Let us denote by $S_X \subset \partial\mathcal{D}_1 \cup \ldots \cup \partial\mathcal{D}_m$ the set of points where X is discontinuous. Note that S_X has measure zero.

The Filippov set-valued map associated with X takes a particularly simple expression for piecewise continuous vector fields, namely,

$$F[X](x) = \text{co}\{\lim_{i \to \infty} X(x_i) \mid x_i \to x, \; x_i \notin S_X\}.$$

This set-valued map can be easily computed as follows. At points of continuity of X, that is, for $x \notin S_X$, we deduce $F[X](x) = \{X(x)\}$, using the consistency property (11). At points of discontinuity of X, that is, for $x \in S_X$, one can prove that $F[X](x)$ is a convex polyhedron in $\mathbb{R}^d$ with vertices of the form

$$X_{|\overline{\mathcal{D}}_k}(x) = \lim_{i \to \infty} X(x_i), \quad \text{with } x_i \to x, \; x_i \in \mathcal{D}_k, \; x_i \notin S_X,$$

for some $k \in \{1, \ldots, m\}$.

As an illustration, let us consider the systems in the section "Examples of discontinuous dynamical systems."

The vector field in the brick sliding on a frictional ramp example is piecewise continuous, with $\mathcal{D}_1 = \{v \in \mathbb{R} \mid v > 0\}$ and $\mathcal{D}_2 = \{v \in \mathbb{R} \mid v < 0\}$. Its associated Filippov set-valued map $F[X] : \mathbb{R} \to \mathbb{R}$,

$$F[X](v) = \begin{cases} \{g(\sin\theta - \nu\cos\theta)\}, & v > 0, \\ \{g(\sin\theta - d\,\nu\cos\theta) \mid d \in [-1, 1]\}, & v = 0, \\ \{g(\sin\theta + \nu\cos\theta)\}, & v < 0, \end{cases}$$

is singleton-valued outside $S_X = \{0\}$, and a closed segment at 0.

The discontinuous "move-away-from-closest-neighbor" vector field for one agent moving in the square $X : [-1, 1]^2 \to \mathbb{R}^2$ is piecewise continuous, with

$$\mathcal{D}_1 = \{(x_1, x_2) \in [-1, 1]^2 \mid -x_1 < x_2 < x_1\}, \qquad \mathcal{D}_2 = \{(x_1, x_2) \in [-1, 1]^2 \mid x_2 < x_1 < -x_2\},$$
$$\mathcal{D}_3 = \{(x_1, x_2) \in [-1, 1]^2 \mid x_1 < x_2 < -x_1\}, \qquad \mathcal{D}_4 = \{(x_1, x_2) \in [-1, 1]^2 \mid -x_2 < x_1 < x_2\}.$$

Its Filippov set-valued map, described in the section "Move-away-from-closest-neighbor interaction law for one agent in a square –revisited," maps points outside $S_X = \{(a, a) \in [-1, 1]^2 \mid a \in [-1, 1]\} \cup \{(a, -a) \in [-1, 1]^2 \mid a \in [-1, 1]\}$ to singletons, points in $S_X \setminus \{(0, 0)\}$ to closed segments, and $(0, 0)$ to a square polygon.

The nonsmooth harmonic oscillator also falls into this category. The vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$, $X(x_1, x_2) = (\text{sign}(x_2), -\text{sign}(x_1))$, is continuous on each one of the quadrants $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4\}$, with

$$\mathcal{D}_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0, \; x_2 > 0\}, \qquad \mathcal{D}_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0, \; x_2 < 0\},$$
$$\mathcal{D}_3 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 < 0, \; x_2 < 0\}, \qquad \mathcal{D}_4 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 < 0, \; x_2 > 0\},$$

and discontinuous on $S_X = \{(x_1, 0) \mid x_1 \in \mathbb{R}\} \cup \{(0, x_2) \mid x_2 \in \mathbb{R}\}$. Therefore, X is piecewise continuous. Its Filippov set-valued map $F[X] : \mathbb{R}^2 \to \mathfrak{B}(\mathbb{R}^2)$ is given by

$$F[X](x_1, x_2) = \begin{cases} \{(\text{sign}(x_2), -\text{sign}(x_1))\}, & x_1 \neq 0 \text{ and } x_2 \neq 0, \\ [-1, 1] \times \{-\text{sign}(x_1)\}, & x_1 \neq 0 \text{ and } x_2 = 0, \\ \{\text{sign}(x_2)\} \times [-1, 1], & x_1 = 0 \text{ and } x_2 \neq 0, \\ [-1, 1] \times [-1, 1], & x_1 = 0 \text{ and } x_2 = 0. \end{cases}$$

Let us now discuss what happens on the points of discontinuity of the vector field $X : \mathbb{R}^d \to \mathbb{R}^d$. Let $x \in S_X$ belong to just two boundary sets, $x \in \partial \mathcal{D}_i \cap \partial \mathcal{D}_j$, for some $i, j \in \{1, \ldots, m\}$. In this case, one can see that $F[X](x) = \text{co}\{X_{|\overline{\mathcal{D}}_i}(x), X_{|\overline{\mathcal{D}}_j}(x)\}$. We consider the following possibilities: (i) if all the vectors belonging to $F[X](x)$ point in the direction of $\mathcal{D}_i$, then any Filippov solution that reaches S_X at x continues its motion in $\mathcal{D}_i$ (see Figure 6(a)); (ii) likewise, if all the vectors belonging to $F[X](x)$ point in the direction of $\mathcal{D}_j$, then any Filippov solution that reaches S_X at x continues its motion in $\mathcal{D}_j$ (see Figure 6(b)); and (iii) however, if a vector belonging to $F[X](x)$ is tangent to S_X, then either Filippov solutions start at x and leave S_X immediately (see Figure 6(c)), or there exists Filippov solutions that reach the set S_X at x, and *stay* in S_X afterward (see Figure 6(d)).

The latter kind of trajectories are called *sliding motions*, since they slide along the boundaries of the sets where the vector field is continuous. This is the type of behavior that we saw in the example of the "move-away-from-closest-neighbor" interaction law. Sliding motions can also occur along points belonging to the intersection of more than two sets in $\mathcal{D}_1, \ldots, \mathcal{D}_m$. The theory

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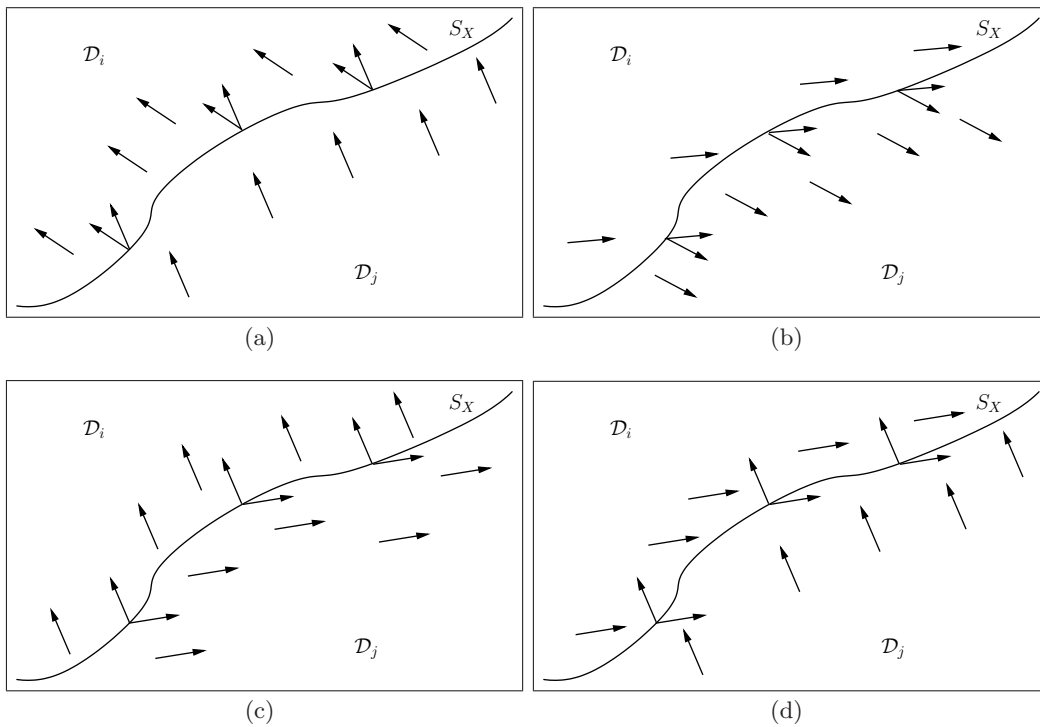**Figure 6.** Piecewise continuous vector fields. The dynamical systems are continuous on $\mathcal{D}_1$ and $\mathcal{D}_2$, and discontinuous at S_X. In cases (a) and (b), Filippov solutions cross the set of discontinuity. In case (c), there are two Filippov solutions starting from points in S_X. Finally, in case (d), Filippov solutions that reach S_X continue its motion sliding along it.

of sliding mode control builds on the existence of this type of trajectories to design stabilizing feedback controllers. These controllers induce sliding surfaces with the right properties in the state space so that the closed-loop system is stable. The interested reader is referred to [29, 30] for a detailed discussion.

The solutions of piecewise continuous vector fields in (i) and (ii) above occur frequently in state-dependent switching dynamical systems. Consider, for instance, the case of two unstable dynamical systems defined on the whole state space. It is conceivable that, by identifying an appropriate switching surface, one can synthesize a stable discontinuous dynamical system on the overall state space. The interested reader may consult [31] and references therein to further explore this topic.

Uniqueness of Filippov solutions

In general, discontinuous dynamical systems do not have unique Filippov solutions. As an example, consider the vector field $X : \mathbb{R} \to \mathbb{R}$, $X(x) = \text{sign}(x)$. For any $x_0 \in \mathbb{R} \setminus \{0\}$, there is a unique Filippov solution starting from x_0. Instead, there are three (maximal) solutions $\gamma_1, \gamma_2, \gamma_3 : [0, \infty) \to \mathbb{R}$ starting from $x(0) = 0$, specifically

$$t \mapsto \gamma_1(t) = -t, \qquad t \mapsto \gamma_2(t) = 0, \qquad t \mapsto \gamma_3(t) = t.$$

The situation depicted in Figure 6(c) is yet another qualitative example where multiple Filippov solutions exist starting from the same initial condition.

In this section, we provide two complementary uniqueness results for Filippov solutions. The first result considers the Filippov set-valued map associated with the discontinuous vector field, and identifies conditions that allow to apply Proposition S2 to the resulting differential inclusion.

Proposition 5. *Let $X : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be measurable and locally essentially bounded. Assume that for all $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, there exist $L_X(t), \varepsilon \in (0, \infty)$ such that for almost every $y, y' \in B(x, \varepsilon)$, one has*

$$(X(t, y) - X(t, y'))^T (y - y') \le L_x(t) \, \|y - y'\|_2^2. \tag{16}$$

Assume that the resulting function $t \mapsto L_X(t)$ is integrable. Then, for any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$, there exists a unique Filippov solution of (1) with initial condition $x(t_0) = x_0$.

Let us apply this result to an example. Consider the vector field $X : \mathbb{R} \to \mathbb{R}$ defined by

$$X(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ -1, & x \notin \mathbb{Q}. \end{cases}$$

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Note that this vector field is discontinuous everywhere in $\mathbb{R}$. The associated Filippov set-valued map $F[X] : \mathbb{R} \to \mathbb{R}$ is $F[X](x) = \{-1\}$ (since $\mathbb{Q}$ has measure zero in $\mathbb{R}$, the value of the vector field at rational points does not play any role in the computation of $F[X]$). Clearly, equation (16) is verified for all $y, y' \notin \mathbb{Q}$. Therefore, there exists a unique solution starting from each initial condition (more precisely, the curve $\gamma : [0, \infty) \to \mathbb{R}$, $t \mapsto \gamma(t) = x(0) - t$).

In general, the Lipschitz-type condition (16) is somewhat restrictive. This assertion is justified by the observation that, in dimension higher than one, piecewise continuous vector fields (arguably, the simpler class of discontinuous vector fields) do not verify the hypotheses of Proposition 5. We carefully explain why in the sidebar "Piecewise Continuous Vector Fields." Figure S1 shows an example of a piecewise continuous vector field with unique solutions starting from each initial condition. However, this uniqueness cannot be guaranteed via Proposition 5.

Next, the following result identifies sufficient conditions for uniqueness specifically tailored for piecewise continuous vector fields.

Proposition 6. *Let* $X : \mathbb{R}^d \to \mathbb{R}^d$ *be a piecewise continuous vector field, with* $\mathbb{R}^d = \mathcal{D}_1 \cup \mathcal{D}_2$. *Let* $S_X = \partial \mathcal{D}_1 = \partial \mathcal{D}_2$ *be the point set where* X *is discontinuous, and assume* S_X *is* C^2 *(that is, around a neighborhood of any of its points, the set can be expressed as the zero level set of twice continuously differentiable functions). Further assume that* $X_{|\overline{\mathcal{D}}_i}$ *is continuously differentiable on* $\mathcal{D}_i$, *for* $i \in \{1, 2\}$, *and that* $X_{|\overline{\mathcal{D}}_1} - X_{|\overline{\mathcal{D}}_2}$ *is continuously differentiable on* S_X. *If for each* $x \in S_X$, *either* $X_{|\overline{\mathcal{D}}_1}(x)$ *points in the direction of* $\mathcal{D}_2$, *or* $X_{|\overline{\mathcal{D}}_2}(x)$ *points in the direction of* $\mathcal{D}_1$, *then there exists a unique Filippov solution of* (1) *starting from each initial condition.*

Note that the hypothesis on X already guarantees uniqueness of solution on each of the domains $\mathcal{D}_1$ and $\mathcal{D}_2$. Roughly speaking, the additional assumptions on X along S_X take care of guaranteeing that uniqueness is not disrupted by the discontinuities. Under the stated assumptions, when reaching S_X, Filippov solutions might cross it or slide along it. Situations like the one depicted in Figure 6(c) are ruled out.

As an application of this result, let us consider the systems in the section "Examples of discontinuous dynamical systems."

For the brick sliding on a frictional ramp example, at $v = 0$, the vector $X_{|\overline{\mathcal{D}}_1}(0)$ points in the direction of $\mathcal{D}_2$, and the vector $X_{|\overline{\mathcal{D}}_2}(0)$ points in the direction of $\mathcal{D}_1$. Proposition 6 then ensures that there exists a unique solution starting from each initial condition;

For the discontinuous vector field for one agent moving in the square $[-1, 1]^2$ under the "move-away-from-closest-neighbor" interaction law, it is convenient to define $\mathcal{D}_5 = \mathcal{D}_1$. Then,

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at any $(x_1, x_2) \in \partial \mathcal{D}_i \cap \partial \mathcal{D}_{i+1} \setminus \{(0,0)\}$, with $i \in \{1, \ldots, 4\}$, the vector $X_{|\overline{\mathcal{D}}_i}(x_1, x_2)$ points in the direction of $\mathcal{D}_{i+1}$, and the vector $X_{|\overline{\mathcal{D}}_{i+1}}(x_1, x_2)$ points in the direction of $\mathcal{D}_i$, see Figure 5. Moreover, there is only one solution (the equilibrium one) starting from $(0,0)$. Therefore, using Proposition 6, we conclude that uniqueness of solutions holds;

For the nonsmooth harmonic oscillator, it is also convenient to define $\mathcal{D}_5 = \mathcal{D}_1$. Then, we can write that, for any $(x_1, x_2) \in \partial \mathcal{D}_i \cap \partial \mathcal{D}_{i+1} \setminus \{(0,0)\}$, with $i \in \{1, \ldots, 4\}$, the vector $X_{|\overline{\mathcal{D}}_i}(x_1, x_2)$ points in the direction of $\mathcal{D}_{i+1}$, see Figure 3(a). Moreover, there is only one solution (the equilibrium one) starting from $(0,0)$. Therefore, using Proposition 6, we conclude that uniqueness of solutions holds.

Proposition 6 can be also applied to piecewise continuous vector fields with an arbitrary number of partitioning domains, provided that set where the vector field is discontinuous is composed of a disjoint union of surfaces resulting from the pairwise intersection of the boundaries of two domains. Other versions of this result can also be stated for non-autonomous piecewise continuous vector fields, and for situations when more than two domains intersect at points of discontinuity. The interested reader is referred to [11, Theorem 4 at page 115].

Solutions of control systems with discontinuous input functions

Let $X : \mathbb{R} \times \mathbb{R}^d \times \mathcal{U} \to \mathbb{R}^d$, with $\mathcal{U} \subset \mathbb{R}^m$ the space of admissible controls, and consider the control equation on $\mathbb{R}^d$,

$$\dot{x}(t) = X(t, x(t), u(t)). \tag{17}$$

At first sight, the most natural way of identifying a notion of solution of this equation would seem to be as follows: select a control input, either open-loop $u : \mathbb{R} \to \mathcal{U}$, closed-loop $u : \mathbb{R}^d \to \mathcal{U}$, or a combination of both $u : \mathbb{R} \times \mathbb{R}^d \to \mathcal{U}$, and then consider the resulting non-autonomous differential equation. In this way, one is back to confronting the question posed above, that is, suitable notions of solution for a discontinuous differential equation.

There are at least a couple of important alternatives to this approach that have been considered in the literature. We discuss them next.

Solutions via differential inclusions

In a similar way as we have done so far, one may associate to the original control system (17) a differential inclusion, and build on it to define the notion of solution. This approach goes as follows: define the set-valued map $G[X] : \mathbb{R} \times \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ by

$$G[X](t, x) = \{X(t, x, u) \mid u \in \mathcal{U}\}.$$

In other words, the set-valued map captures all the directions in $\mathbb{R}^d$ that can be generated with controls belonging to $\mathcal{U}$. Consider now the differential inclusion

$$\dot{x}(t) \in G[X](t, x(t)). \tag{18}$$

A *solution of* (17) defined on $[t_0, t_1] \subset \mathbb{R}$ is a solution of the differential inclusion (18), that is, an absolutely continuous map $\gamma : [t_0, t_1] \to \mathbb{R}^d$ such that $\dot{\gamma}(t) \in G[X](t, \gamma(t))$ for almost every $t \in [t_0, t_1]$.

Clearly, given $u : \mathbb{R} \to \mathcal{U}$, any Caratheodory solution of the control system is also a solution of the associated differential inclusion. Alternatively, one can show [11] that, if X is continuous and $\mathcal{U}$ is compact, the converse is also true. Considering the differential inclusion has the advantage of not focusing the attention on any particular control input, and therefore allows to comprehensively study and understand the properties of the control system as a whole.

Sample-and-hold solutions

Here we introduce the notion of sample-and-hold solution for control systems [32]. As we see later, this notion plays a key role in the stabilization question for asymptotically controllable systems.

A *partition* of the interval $[t_0, t_1]$ is a strictly increasing sequence $\pi = \{s_0 = t_0 < s_1 < \cdots < s_N = t_1\}$. Note that the partition does not need to be finite, and that one can define the notion of partition of $[t_0, \infty)$ similarly. The diameter of π is $\mathrm{diam}(\pi) = \sup\{s_i - s_{i-1} \mid i \in \{1, \ldots, N\}\}$. Given a feedback law $u : \mathbb{R} \times \mathbb{R}^d \to \mathcal{U}$ and a partition π of $[t_0, t_1]$, a π-*solution of* (17) defined on $[t_0, t_1] \subset \mathbb{R}$ is the map $\gamma : [t_0, t_1] \to \mathbb{R}^d$ recursively defined as follows: for $i \in \{1, \ldots, N-1\}$, the curve $[t_{i-1}, t_i] \ni t \mapsto \gamma(t)$ is a Caratheodory solution of the differential equation

$$\dot{x}(t) = X(t, x(t), u(t_{i-1}, x(t_{i-1}))).$$

Roughly speaking, the control is held fixed throughout each interval of the partition at the value corresponding to the state at the beginning of the interval, and then the corresponding differential equation is solved, which explains why π-solutions are also referred to as *sample-and-hold* solutions. From our previous discussion on Caratheodory solutions, it is not difficult to derive conditions on the control system for the existence of π-solutions. Indeed, existence of π-solutions is guaranteed

if (i) for all $u \in \mathcal{U} \subset \mathbb{R}^m$ and almost all $t \in \mathbb{R}$, the map $x \mapsto X(t, x, u)$ is continuous, (ii) for all $u \in \mathcal{U} \subset \mathbb{R}^m$ and all $x \in \mathbb{R}^d$, the map $t \mapsto X(t, x, u)$ is measurable, and (iii) for all $u \in \mathcal{U} \subset \mathbb{R}^m$, $(t, x) \to X(t, x, u)$ is locally essentially bounded.

Generalized sample-and-hold solutions of (17) are defined in [9] as the uniform limit of a sequence of π-solutions of (17) as $\text{diam}(\pi) \to 0$. Interestingly, in general, generalized sample-and-hold solutions are not Caratheodory solutions, although conditions exist under which the inclusion holds, see [9].

Nonsmooth Analysis

It should come at no surprise to the reader that, if we have "gone discontinuous" with differential equations, we now "go nonsmooth" with the candidate Lyapunov functions. When examining the stability properties of discontinuous differential equations and differential inclusions, there are additional reasons to consider nonsmooth Lyapunov functions. The nonsmooth harmonic oscillator is a good example of what we mean, because it does not admit any smooth Lyapunov function. To see why, recall that all the Filippov solutions of the discontinuous system are periodic (see Figure 3). If such a smooth function exists, it necessarily has to be constant on each diamond. Therefore, since the level sets of the function are necessarily one-dimensional, each diamond would be a level set, which contradicts the fact that the function is smooth. This observation, taken from [9], is a simple illustration that our efforts to consider nonsmooth Lyapunov functions when considering discontinuous dynamics are not gratuitous.

In this section we discuss two tools from nonsmooth analysis: generalized gradients and proximal subdifferentials, see for instance [13, 33]. As with the concept of solution of discontinuous differential equations, the literature is full of generalized derivative notions for the case when a function fails to be differentiable. These notions include, in addition to the two considered in this section, generalized (super or sub) differentials, (upper or lower, right or left) Dini derivatives, and contingent derivatives. The interested reader is referred to [13, 15, 34, 35] and references therein for a complete account. Here, we have chosen to focus on the notions of generalized gradients and proximal subdifferentials because of their important role on providing applicable stability tools for discontinuous differential equations. The functions considered here are always defined on a finite-dimensional Euclidean space, but we note that these objects are actually well-defined in Banach and Hilbert spaces.

Generalized gradients of locally Lipschitz functions

From Rademacher's Theorem [33], locally Lipschitz functions are differentiable almost everywhere (in the sense of Lebesgue measure). When considering a locally Lipschitz function as a candidate Lyapunov function, this statement may rise the following question: if the gradient of a locally Lipschitz function exists almost everywhere, should we really care for those points where it does not exist? Conceivably, the solutions of the dynamical systems under study stay almost everywhere away from the "bad" points where we do not have any gradient of the function. However, such assumption turns out not to be true in general. As we show later, there are cases when the solutions of the dynamical system insist on staying on the "bad" points forever. In that case, having some sort of gradient information is helpful.

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function. If Ω_f denotes the set of points in $\mathbb{R}^d$ at which f fails to be differentiable, and S denotes any set of measure zero, the *generalized gradient* $\partial f : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ of f is defined by

$$\partial f(x) = \mathrm{co}\{\lim_{i \to \infty} \nabla f(x_i) \mid x_i \to x, \ x_i \notin S \cup \Omega_f\}.$$

From the definition, the generalized gradient at a point provides convex combinations of all the possible limits of the gradient at neighboring points (where the function is in fact differentiable). Note that this definition coincides with $\nabla f(x)$ when f is continuously differentiable at x. Other equivalent definitions of the generalized gradient can be found in [33].

Let us compute the generalized gradient in a particular case. Consider the locally Lipschitz function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$. The function is differentiable everywhere except for 0. Actually, $\nabla f(x) = 1$ for $x > 0$ and $\nabla f(x) = -1$ for $x < 0$. Therefore, we deduce

$$\partial f(0) = \mathrm{co}\{1, -1\} = [-1, 1].$$

Computing the generalized gradient

As one might imagine, the computation of the generalized gradient of a locally Lipschitz function is not an easy task in general. In addition to the "brute force" approach, there are a number of results that can help us compute it. Many of the standard results that are valid for usual derivatives have their counterpart in this setting. We summarize some of them here, and refer the reader to [13, 33] for a complete exposition.

Dilation rule. For $f : \mathbb{R}^d \to \mathbb{R}$ locally Lipschitz at $x \in \mathbb{R}^d$ and $s \in \mathbb{R}$, the function sf is locally Lipschitz at x, and

$$\partial(sf)(x) = s \, \partial f(x). \tag{19}$$

Sum rule. For $f_1, f_2 : \mathbb{R}^d \to \mathbb{R}$ locally Lipschitz at $x \in \mathbb{R}^d$, and any scalars $s_1, s_2 \in \mathbb{R}$, the function $s_1 f_1 + s_2 f_2$ is locally Lipschitz at x, and

$$\partial\big(s_1 f_1 + s_2 f_2\big)(x) \subset s_1 \partial f_1(x) + s_2 \partial f_2(x). \tag{20}$$

Moreover, if f_1 and f_2 are regular at x, and $s_1, s_2 \in [0, \infty)$, then equality holds and $s_1 f_1 + s_2 f_2$ is regular at x.

Product rule. For $f_1, f_2 : \mathbb{R}^d \to \mathbb{R}$ locally Lipschitz at $x \in \mathbb{R}^d$, the function $f_1 f_2$ is locally Lipschitz at x, and

$$\partial\big(f_1 f_2\big)(x) \subset f_2(x) \partial f_1(x) + f_1(x) \partial f_2(x). \tag{21}$$

Moreover, if f_1 and f_2 are regular at x, and $f_1(x), f_2(x) \geq 0$, then equality holds and $f_1 f_2$ is regular at x.

Quotient rule. For $f_1, f_2 : \mathbb{R}^d \to \mathbb{R}$ locally Lipschitz at $x \in \mathbb{R}^d$, with $f_2(x) \neq 0$, the function f_1/f_2 is locally Lipschitz at x, and

$$\partial\Big(\frac{f_1}{f_2}\Big)(x) \subset \frac{f_2(x) \partial f_1(x) - f_1(x) \partial f_2(x)}{f_2^2(x)}. \tag{22}$$

Moreover, if f_1 and $-f_2$ are regular at x, and $f_1(x) \geq 0, f_2(x) > 0$, then equality holds and f_1/f_2 is regular at x.

Chain rule. For $h : \mathbb{R}^d \to \mathbb{R}^m$, with each component locally Lipschitz at $x \in \mathbb{R}^d$, and $g : \mathbb{R}^m \to \mathbb{R}$ locally Lipschitz at $h(x) \in \mathbb{R}^m$, the function $g \circ h$ is locally Lipschitz at x, and

$$\partial\big(g \circ h\big)(x) \subset \mathrm{co}\Big\{ \sum_{k=1}^m \alpha_k \zeta_k \mid (\alpha_1, \ldots, \alpha_m) \in \partial g(h(x)), \; (\zeta_1, \ldots, \zeta_m) \in \partial h_1(x) \times \cdots \times \partial h_m(x) \Big\}. \tag{23}$$

Moreover, if g is regular at $h(x)$, each component of h is regular at x, and every element of $\partial g(h(x))$ belongs to $[0, \infty)^d$, then equality holds and $g \circ h$ is regular at x.

Let us highlight here a particularly useful result from [33, Proposition 2.3.12] concerning the generalized gradient of max and min functions.

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Proposition 7. *Let* $f_k : \mathbb{R}^d \to \mathbb{R}$, $k \in \{1, \ldots, m\}$ *be locally Lipschitz functions at* $x \in \mathbb{R}^d$ *and consider the functions*

$$f_{\max}(x') = \max\{f_k(x') \mid k \in \{1, \ldots, m\}\}, \qquad f_{\min}(x') = \min\{f_k(x') \mid k \in \{1, \ldots, m\}\}.$$

Then,

(i) $f_{\max}$ *and* $f_{\min}$ *are locally Lipschitz at* x,

(ii) *if* $I_{\max}(x')$ *denotes the set of indexes* k *for which* $f_k(x') = f_{\max}(x')$*, we have*

$$\partial f_{\max}(x) \subset \mathrm{co}\{\partial f_i(x) \mid i \in I_{\max}(x)\}, \tag{24}$$

and if f_i, $i \in I_{\max}(x)$*, is regular at* x*, then equality holds and* $f_{\max}$ *is regular at* x,

(iii) *if* $I_{\min}(x')$ *denotes the set of indexes* k *for which* $f_k(x') = f_{\min}(x')$*, we have*

$$\partial f_{\min}(x) \subset \mathrm{co}\{\partial f_i(x) \mid i \in I_{\min}(x)\}, \tag{25}$$

and if $-f_i$, $i \in I_{\min}(x)$*, is regular at* x*, then equality holds and* $-f_{\min}$ *is regular at* x.

As a consequence of this result, the maximum of a finite set of continuously differentiable functions is a locally Lipschitz and regular function, and its generalized gradient is easily computable at each point as the convex closure of the gradients of the functions that attain the maximum at that particular point. As an example, the function $f_1(x) = |x|$ can be re-written as $f_1(x) = \max\{x, -x\}$. Both $x \mapsto x$ and $x \mapsto -x$ are continuously differentiable, and hence locally Lipschitz and regular. Therefore, according to Proposition 7(i) and (ii), so is f_1, and its generalized gradient is

$$\partial f_1(x) = \begin{cases} \{1\}, & x > 0, \\ [-1, 1], & x = 0, \\ \{-1\}, & x < 0, \end{cases} \tag{26}$$

which is the same result that we obtained earlier by direct computation.

Note that the minimum of a finite set of regular functions is in general not regular. A simple example is given by $f_2(x) = \min\{x, -x\} = -|x|$, which is not regular at 0, as we showed in the sidebar "Regular Functions." However, according to Proposition 7(i) and (iii), this fact does not mean that its generalized gradient cannot be computed. Indeed, one has

$$\partial f_2(x) = \begin{cases} \{-1\}, & x > 0, \\ [-1, 1], & x = 0, \\ \{1\}, & x < 0. \end{cases} \tag{27}$$

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Critical points and directions of descent

A *critical* point of $f : \mathbb{R}^d \to \mathbb{R}$ is a point $x \in \mathbb{R}^d$ such that $0 \in \partial f(x)$. The maxima and minima of locally Lipschitz functions are critical points according to this definition. As an example, $x = 0$ is a minimum of $f(x) = |x|$, and indeed one verifies that $0 \in \partial f(0)$.

When the function f is continuously differentiable, the gradient ∇f provides the direction of maximum ascent (respectively, $-\nabla f$ provides the direction of maximum descent). When considering locally Lipschitz functions, however, one faces the following question: given that the generalized gradient is a set of directions, rather than a single one, which one are the right ones to choose? Without loss of generality, we restrict our discussion to directions of descent, since a direction of descent of $-f$ corresponds to a direction of ascent of f, and f is locally Lipschitz if and only if $-f$ is locally Lipschitz.

Let $\mathrm{Ln} : \mathfrak{B}(\mathbb{R}^d) \to \mathfrak{B}(\mathbb{R}^d)$ be the set-valued map that associates to each subset S of $\mathbb{R}^d$ the set of least-norm elements of its closure $\overline{S}$. If the set S is convex, then the set $\mathrm{Ln}(S)$ reduces to a singleton and we note the equivalence $\mathrm{Ln}(S) = \mathrm{proj}_S(0)$. For a locally Lipschitz function f, consider the *generalized gradient vector field* $\mathrm{Ln}(\partial f) : \mathbb{R}^d \to \mathbb{R}^d$

$$x \mapsto \mathrm{Ln}(\partial f)(x) = \mathrm{Ln}(\partial f(x)).$$

It turns out that $\mathrm{Ln}(\partial f)(x)$ is a direction of descent at $x \in \mathbb{R}^d$. More precisely, following [33], one finds that if $0 \notin \partial f(x)$, then there exists $T > 0$ such that

$$f(x - t\,\mathrm{Ln}(\partial f)(x)) \leq f(x) - \frac{t}{2}\| \mathrm{Ln}(\partial f)(x)\|_2^2, \quad 0 < t < T. \tag{28}$$

Minimum distance to polygonal boundary

Let $Q \subset \mathbb{R}^2$ be a convex polygon. Consider the minimum distance function $\mathrm{sm}_Q : Q \to \mathbb{R}$ from any point within the polygon to its boundary defined by

$$\mathrm{sm}_Q(p) = \min\{\|p - q\|_2 \mid q \in \partial Q\}.$$

Note that the value of sm_Q corresponds to the radius of the largest disk contained in the polygon with center p. Moreover, this function is locally Lipschitz on Q. To show this, simply rewrite the function as

$$\mathrm{sm}_Q(p) = \min\{\mathrm{dist}(p, e) \mid e \text{ edge of } Q\},$$

where $\mathrm{dist}(p, e)$ denotes the Euclidean distance from the point p to the edge e. Let us consider the generalized gradient vector field corresponding to this function (if one prefers to have a function defined on the whole space, as we have been using in this section, one can easily extend the definition of sm_Q outside Q by setting $\mathrm{sm}_Q(p) = -\min\{\|p - q\|_2 \mid q \in \partial Q\}$ for $p \notin Q$, and proceed with the discussion). Applying Proposition 7(iii), we deduce that $-\mathrm{sm}_Q$ is regular on Q and its generalized gradient is

$$\partial\,\mathrm{sm}_Q(p) = \mathrm{co}\{\mathrm{n}_e \mid e \text{ edge of } Q \text{ such that } \mathrm{sm}_Q(p) = \mathrm{dist}(p, e)\}, \quad p \in Q,$$

where n_e denotes the unit normal to the edge e pointing toward the interior of Q. Therefore, at points p in Q where there is a unique edge e of Q which is closest to p, the function sm_Q is differentiable, and its generalized gradient vector field is given by $\mathrm{Ln}(\mathrm{sm}_Q)(p) = \mathrm{n}_e$. Note that this vector field corresponds to the "move-away-from-closest-neighbor" interaction law for one agent moving in the polygon! At points p of Q where various edges $\{e_1, \ldots, e_m\}$ are at the same minimum distance to p, the function sm_Q is not differentiable, and its generalized gradient vector field is given by the least-norm element in $\mathrm{co}\{\mathrm{n}_{e_1}, \ldots, \mathrm{n}_{e_m}\}$. If p is not a critical point, 0 does not belong to the latter set, and the least-norm element points in the direction of the bisector line between two of the edges in $\{e_1, \ldots, e_m\}$. Figure 7 shows a plot of the generalized gradient vector field of sm_Q on the square $Q = [-1, 1]^2$. Note the similarity with the plot in Figure 5.
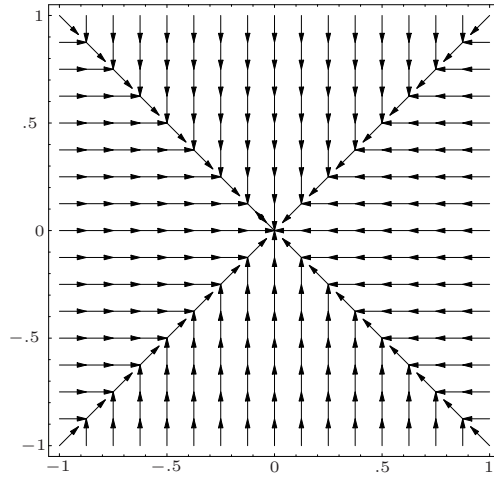

Figure 7. Generalized gradient vector field. The plot shows the generalized gradient vector field of the minimum distance to polygonal boundary function $\mathrm{sm}_Q : Q \to \mathbb{R}$ **on the square** $[-1, 1]^2$. **The vector field is discontinuous on the diagonals of the square.**

Indeed, one can characterize [36] the critical points of sm_Q as

$$0 \in \partial\,\mathrm{sm}_Q(p) \text{ if and only if } p \text{ belongs to the incenter set of } Q.$$

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The incenter set of Q is composed of the centers of the largest-radius disks contained in Q. In general, the incenter set is not a singleton (think, for instance, of a rectangle), but a segment. However, one can also show that if $0 \in \text{interior}(\partial \, \text{sm}_Q(p))$, then the incenter set of Q is the singleton $\{p\}$.

Nonsmooth gradient flows

Given a locally Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$, one can define the nonsmooth analog of the classical gradient flow of a differentiable function as

$$\dot{x}(t) = -\text{Ln}(\partial f)(x(t)). \tag{29}$$

According to (28), unless the flow is already at a critical point, $-\text{Ln}(\partial f)(x)$ is always a direction of descent at x. Note that this nonsmooth gradient vector field is discontinuous, and therefore we have to specify the notion of solution that we consider. In this case, we select the Filippov notion. Since f is locally Lipschitz, $\text{Ln}(\partial f) = \nabla f$ almost everywhere. A remarkable fact [28] is that the Filippov set-valued map associated with the nonsmooth gradient flow of f is precisely the generalized gradient of the function, that is,

Filippov set-valued map of nonsmooth gradient. For $f : \mathbb{R}^d \to \mathbb{R}$ locally Lipschitz, the Filippov set-valued map $F[\text{Ln}(\partial f)] : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ of the nonsmooth gradient of f is equal to the generalized gradient $\partial f : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ of f,

$$F[\text{Ln}(\partial f)](x) = \partial f(x), \quad x \in \mathbb{R}^d.$$

As a consequence of this result, note that the discontinuous system (29) is equivalent to the differential inclusion

$$\dot{x}(t) \in -\partial f(x(t)).$$

How can we analyze the asymptotic behavior of the trajectories of this system? When the function f is differentiable, the LaSalle Invariance Principle allows us to deduce that, for functions with bounded level sets, the trajectories of the gradient flow asymptotically converge to the set of critical points. The key tool behind this result is being able to establish that the value of the function decreases along the trajectories of the system. This behavior is formally expressed through the notion of Lie derivative. We discuss later suitable generalizations of the notion of Lie derivative to the nonsmooth case. These notions allow us, among other things, to study the asymptotic convergence properties of the trajectories of nonsmooth gradient flows.

Proximal subdifferentials of lower semicontinuous functions

A complementary set of nonsmooth analysis tools to deal with Lyapunov functions is given by proximal subdifferentials. This concept has the advantage of being defined for a larger class of functions, namely, lower semicontinuous (instead of locally Lipschitz) functions. Generalized gradients provide us with directional descent information, that is, directions along which the function decreases. The price we pay by using proximal subdifferentials is that explicit descent directions are in generally not known to us. Proximal subdifferentials, however, still allow us to reason about the monotonic properties of the function, which as we show later, turns out to be sufficient to provide stability results.

A function $f : \mathbb{R}^d \to \mathbb{R}$ is *lower semicontinuous* at $x \in \mathbb{R}^d$ if for all $\varepsilon \in (0, \infty)$, there exists $\delta \in (0, \infty)$ such that $f(y) \geq f(x) - \varepsilon$, for $y \in B(x, \delta)$. In this article, we restrict our attention to real-valued lower semicontinuous functions. Lower semicontinuous functions with extended real values are considered in [13]. A function $f : \mathbb{R}^d \to \mathbb{R}$ is *upper semicontinuous* at $x \in \mathbb{R}^d$ if $-f$ is lower semicontinuous at x. Note that f is continuous at x if and only if f is both upper and lower semicontinuous at x. For a lower semicontinuous function $f : \mathbb{R}^d \to \mathbb{R}$, a vector $\zeta \in \mathbb{R}^d$ is a *proximal subgradient of f at $x \in \mathbb{R}^d$* if there exist $\sigma, \delta \in (0, \infty)$ such that for all $y \in B(x, \delta)$,

$$f(y) \geq f(x) + \zeta(y - x) - \sigma^2 \|y - x\|_2^2. \tag{30}$$

The set of all proximal subgradients of f at x is the *proximal subdifferential of f at x*, and denoted $\partial_P f(x)$. The proximal subdifferential at x is always convex. However, it is not necessarily open, closed, bounded or nonempty. Geometrically, the definition of proximal subgradient can be interpreted as follows. Equation (30) is equivalent to saying that, around x, the function $y \mapsto f(y)$ majorizes the quadratic function $y \mapsto f(x) + \zeta(y - x) - \sigma^2 \|y - x\|_2^2$. In other words, there exists a parabola that locally fits under the graph of f at $(x, f(x))$. This geometric interpretation is indeed very useful for the explicit computation of the proximal subdifferential.

Let us compute the proximal subdifferential in two particular cases. Consider the locally Lipschitz functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = |x|$ and $f_2(x) = -|x|$. Using the geometric interpretation of (30), it is not difficult to see that

$$\partial_P f_1(x) = \begin{cases} \{1\}, & x < 0, \\ [-1, 1], & x = 0, \\ \{-1\}, & x > 0, \end{cases} \qquad \partial_P f_2(x) = \begin{cases} \{-1\}, & x < 0, \\ \emptyset, & x = 0, \\ \{1\}, & x > 0. \end{cases}$$

Compare this result with the generalized gradients of f_1 in (26) and of f_2 in (27).

Unlike the case of generalized gradients, the proximal subdifferential may not coincide with $\nabla f(x)$ when f is continuously differentiable. The function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto -|x|^{3/2}$, is

continuously differentiable, but $\partial_P f(0) = \emptyset$. In fact, [37] provides an example of a continuously differentiable function on $\mathbb{R}$ which has an empty proximal subdifferential almost everywhere. However, it should be noted that the density theorem (cf. [13, Theorem 3.1]) states that the proximal subdifferential of a lower semicontinuous function is always nonempty in a dense set of its domain of definition.

On the other hand, the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt{|x|}$ provides an example where proximal subdifferentials are more useful than generalized gradients. The function is continuous at 0, but not locally Lipschitz at 0, which precisely corresponds to its global minimum. Hence the generalized gradient does not help us here. The function is lower semicontinuous, and has a well-defined proximal subdifferential,

$$\partial_P f(x) = \begin{cases} \left\{\frac{1}{2}\frac{1}{\sqrt{x}}\right\}, & x > 0, \\ \mathbb{R}, & x = 0, \\ \left\{-\frac{1}{2}\frac{1}{\sqrt{-x}}\right\}, & x < 0. \end{cases}$$

If $f : \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz at $x \in \mathbb{R}^d$, then the proximal subdifferential of f at x is bounded. In general, the relationship between the generalized gradient and the proximal subdifferential of a function f locally Lipschitz at $x \in \mathbb{R}^d$ is expressed by

$$\partial f(x) = \mathrm{co}\{ \lim_{n \to \infty} \zeta_n \in \mathbb{R}^d \mid \zeta_n \in \partial_P f(x_n) \text{ and } \lim_{n \to \infty} x_n = x\}.$$

Computing the proximal subdifferential

As with the generalized gradient, the computation of the proximal subdifferential gradient of a lower semicontinuous function is not straightforward in general. Here we provide some useful results following the exposition in [13].

Dilation rule. For $f : \mathbb{R}^d \to \mathbb{R}$ lower semicontinuous at $x \in \mathbb{R}^d$ and $s \in (0, \infty)$, the function sf is lower semicontinuous at x, and

$$\partial_P(sf)(x) = s\,\partial_P f(x). \tag{31}$$

Sum rule. For $f_1, f_2 : \mathbb{R}^d \to \mathbb{R}$ lower semicontinuous at $x \in \mathbb{R}^d$, the function $f_1 + f_2$ is lower semicontinuous at x, and

$$\partial_P f_1(x) + \partial_P f_2(x) \subset \partial_P(f_1 + f_2)(x). \tag{32}$$

Moreover, if either f_1 or f_2 are twice continuously differentiable, then equality holds.

Chain rule. For either $h : \mathbb{R}^d \to \mathbb{R}^m$ linear and $g : \mathbb{R}^m \to \mathbb{R}$ lower semicontinuous at $h(x) \in \mathbb{R}^m$, or $h : \mathbb{R}^d \to \mathbb{R}^m$ locally Lipschitz at $x \in \mathbb{R}^d$ and $g : \mathbb{R}^m \to \mathbb{R}$ locally Lipschitz at $h(x) \in \mathbb{R}^m$, the following holds: for $\zeta \in \partial_P(g \circ h)(x)$ and any $\varepsilon \in (0, \infty)$, there exist $\tilde{x} \in \mathbb{R}^d$, $\tilde{y} \in \mathbb{R}^m$, and $\gamma \in \partial_P g(\tilde{y})$ with $\max\{\|\tilde{x} - x\|_2, \|\tilde{y} - h(x)\|_2\} < \varepsilon$ such that $\|h(\tilde{x}) - h(x)\|_2 < \varepsilon$ and

$$\zeta \in \partial_P(\langle \gamma, h(\cdot) \rangle)(\tilde{x}) + \varepsilon B(0, 1). \tag{33}$$

The statement of the chain rule above shows one of the characteristic features when dealing with proximal subdifferentials: in many occasions, arguments and results are expressed with objects evaluated at points in a neighborhood of the specific point of interest, rather than at the point itself.

The computation of the proximal subdifferential of twice continuously differentiable functions is particularly simple. For $f : \mathbb{R}^d \to \mathbb{R}$ twice continuously differentiable on $U \subset \mathbb{R}^d$ open, one has

$$\partial_P f(x) = \{\nabla f(x)\}, \quad \text{for all } x \in U. \tag{34}$$

This simplicity also works for continuously differentiable convex functions, as the following result states.

Proposition 8. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be lower semicontinuous and convex, and let $x \in \mathbb{R}^d$. Then,*

(i) *$\zeta \in \partial_P f(x)$ if and only if $f(y) \geq f(x) + \zeta(y - x)$, for all $y \in \mathbb{R}^d$;*

(ii) *the map $x \mapsto \partial_P f(x)$ takes nonempty, compact and convex values, and is upper semicontinuous and locally bounded;*

(iii) *if, in addition, f is continuously differentiable, then $\partial_P f(x) = \{\nabla f(x)\}$, for all $x \in \mathbb{R}^d$;*

Regarding critical points, if x is a local minimum of a lower semicontinuous function $f : \mathbb{R}^d \to \mathbb{R}$, then $0 \in \partial_P f(x)$. Conversely, if f is lower semicontinuous and convex, and $0 \in \partial_P f(x)$, then x is a global minimum of f. If one is interested in maxima, then instead of the notions of lower semicontinuous functions, convex functions and proximal subdifferentials, one needs to consider upper semicontinuous functions, concave functions and proximal superdifferentials, respectively (see [13]).

Gradient differential inclusions

In general, one cannot define a nonsmooth gradient flow associated to a lower semicontinuous function, because, as we have observed above, the proximal subdifferential might be empty almost everywhere. However, following Proposition 8(ii), we can associate a nonsmooth gradient flow to functions that are lower semicontinuous and convex, as we briefly discuss next following [38].

Let $f : \mathbb{R}^d \to \mathbb{R}$ be lower semicontinuous and convex. Consider the gradient differential inclusion

$$\dot{x}(t) \in -\partial_P f(x(t)). \tag{35}$$

Using the properties of the proximal subdifferential stated in Proposition 8(ii), existence of solutions of this differential inclusion is guaranteed by Proposition S1. Moreover, uniqueness of solutions can also be established. To show this, let $x, y \in \mathbb{R}^d$, and take $\zeta_1 \in -\partial_P f(x)$ and $\zeta_2 \in -\partial_P f(y)$. Using Proposition 8(i), we have

$$f(y) \geq f(x) - \zeta_1(y - x), \qquad f(x) \geq f(y) - \zeta_2(x - y).$$

From here, we deduce $-\zeta_1(y - x) \leq f(y) - f(x) \leq -\zeta_2(y - x)$, and therefore $(\zeta_2 - \zeta_1)(y - x) \leq 0$, which, in particular, implies that the set-valued map $x \mapsto -\partial_P f(x)$ verifies the one-sided Lipschitz condition (S2). Proposition S2 guarantees then uniqueness of solutions.

Once we know that solutions exist and are unique, the next natural question is to understand their asymptotic behavior. To analyze it, we need to introduce tools specifically tailored for this nonsmooth setting that allow us to establish, among other things, the monotonic behavior of the function f along the solutions. We tackle this task in the next two sections.

Nonsmooth Stability Analysis

In this section, we present tools to study the stability properties of discontinuous dynamical systems. Unless explicitly mentioned otherwise, the stability notions employed here correspond to the usual ones for differential equations, see, for instance [39]. The presentation of the results focuses on the setup of autonomous differential inclusions,

$$\dot{x}(t) \in \mathcal{F}(x(t)), \tag{36}$$

where $\mathcal{F} : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$. Throughout the section, we assume that the set-valued map $\mathcal{F}$ verifies the hypothesis of Proposition S1, so that the existence of solutions of the differential inclusion is

guaranteed. From our previous discussion, it should be clear that the setup of differential inclusions has a direct application to the scenario of discontinuous differential equations and control systems. The results presented here can be easily made explicit for the notions of solution introduced earlier (for instance, for Filippov solutions, by taking $\mathcal{F} = F[X]$), and we leave this task to the reader.

Before proceeding with our exposition, let us make a couple of remarks. The first one concerns the wordings "strong" and "weak." As we already observed, solutions of discontinuous systems are generally not unique. Therefore, when considering properties such as Lyapunov stability or invariance, one needs to specify if one is paying attention to *a* particular solution starting from an initial condition ("weak") or to *all* the solutions starting from an initial condition ("strong"). As an example, a set $M \subset \mathbb{R}^d$ is *weakly invariant* for (36) if for each $x_0 \in M$, M contains *a* maximal solution of (36) with initial condition x_0. Similarly, $M \subset \mathbb{R}^d$ is *strongly invariant* for (36) if for each $x_0 \in M$, M contains *all* maximal solutions of (36) with initial condition x_0.

The second remark concerns the notion of limit point of solutions of the differential inclusion. A point $x \in \mathbb{R}^d$ is a *limit point* of a solution γ of (36) if there exists a sequence $\{t_n\}_{n \in \mathbb{N}}$ such that $\gamma(t_n) \to x$ as $n \to \infty$. We denote by $\Omega(\gamma)$ the set of limit points of γ. Under the hypothesis of Proposition S1, $\Omega(\gamma)$ is a weakly invariant set. Moreover, if the solution γ lies in a bounded domain, then $\Omega(\gamma)$ is nonempty, bounded, connected, and $\gamma(t) \to \Omega(\gamma)$ as $t \to \infty$, see [11].

Stability analysis via generalized gradients of nonsmooth Lyapunov functions

In this section, we discuss nonsmooth stability analysis results that invoke locally Lipschitz functions and their generalized gradients. We have chosen a number of important results taken from different sources in the literature. The discussion presented here does not intend to be a comprehensive account of such a vast topic, but rather serve as a motivation to further explore it. The interested reader may consult the books [3, 11] and the journal papers [25, 40, 41] for further reference.

Lie derivatives and monotonicity

A common theme in stability analysis is the possibility of establishing the monotonic evolution along the trajectories of the system of a candidate Lyapunov function. Mathematically, the evolution of a function along trajectories is captured by the notion of Lie derivative. Our first task here is then to generalize this notion to the setup of discontinuous systems following [25], see also [40, 41].

Given a locally Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ and a set-valued map $\mathcal{F} : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$, the *set-valued Lie derivative* $\widetilde{\mathcal{L}}_\mathcal{F} f : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R})$ of f with respect to $\mathcal{F}$ at x is defined as

$$\widetilde{\mathcal{L}}_\mathcal{F} f(x) = \{a \in \mathbb{R} \mid \text{there exists } v \in \mathcal{F}(x) \text{ such that } \zeta^T v = a, \text{ for all } \zeta \in \partial f(x)\}. \tag{37}$$

If $\mathcal{F}$ takes convex and compact values, for each $x \in \mathbb{R}^d$, $\widetilde{\mathcal{L}}_\mathcal{F} f(x)$ is a closed and bounded interval in $\mathbb{R}$, possibly empty. If f is continuously differentiable at x, then $\widetilde{\mathcal{L}}_\mathcal{F} f(x) = \{(\nabla f)^T v \mid v \in \mathcal{F}(x)\}$. The importance of the set-valued Lie derivative stems from the fact that it allows us to study how the function f evolves along the solutions of a differential inclusion without having to obtain them in closed form. Specifically, we have the following result.

Proposition 9. *Let $\gamma : [t_0, t_1] \to \mathbb{R}^d$ be a solution of the differential inclusion* (36), *and let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz and regular function. Then*

(i) *the composition $t \mapsto f(\gamma(t))$ is differentiable at almost every $t \in [t_0, t_1]$, and*

(ii) *the derivative of $t \mapsto f(\gamma(t))$ verifies*

$$\frac{d}{dt}\big(f(\gamma(t))\big) \in \widetilde{\mathcal{L}}_\mathcal{F} f(\gamma(t)) \quad \textit{for almost every } t \in [t_0, t_1]. \tag{38}$$

Given a discontinuous vector field $X : \mathbb{R}^d \to \mathbb{R}^d$, consider the solutions of (1) in the Filippov sense. In that case, with a little abuse of notation, we denote $\widetilde{\mathcal{L}}_X f = \widetilde{\mathcal{L}}_{F[X]} f$. Note that if X is continuous at x, then $F[X](x) = \{X(x)\}$, and therefore, $\widetilde{\mathcal{L}}_X f(x)$ corresponds to the singleton $\{\mathcal{L}_X f(x)\}$, the usual Lie derivative of f in the direction of X at x.

Let us illustrate the importance of this result in an example.

Monotonicity in the nonsmooth harmonic oscillator

For the nonsmooth harmonic oscillator, consider the locally Lipschitz and regular map $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = |x_1| + |x_2|$ (recall that Figure 3(b) shows the contour plot of f). Let us determine how the function evolves along the solutions of the dynamical system by looking at the set-valued Lie derivative. First, we compute the generalized gradient of f. To do so, we rewrite the function as $f(x_1, x_2) = \max\{x_1, -x_1\} + \max\{x_2, -x_2\}$, and apply Proposition 7(ii) and the sum rule to find

$$\partial f(x_1, x_2) = \begin{cases} \{(\text{sign}(x_1), \text{sign}(x_2))\}, & x_1 \neq 0 \text{ and } x_2 \neq 0, \\ \{\text{sign}(x_1)\} \times [-1, 1], & x_1 \neq 0 \text{ and } x_2 = 0, \\ [-1, 1] \times \{\text{sign}(x_2)\}, & x_1 = 0 \text{ and } x_2 \neq 0, \\ [-1, 1] \times [-1, 1], & x_1 = 0 \text{ and } x_2 = 0. \end{cases}$$

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With this information, we are ready to compute the set-valued Lie derivative $\widetilde{\mathcal{L}}_X f : \mathbb{R}^2 \to \mathfrak{B}(\mathbb{R})$ as

$$\widetilde{\mathcal{L}}_X f(x_1, x_2) = \begin{cases} \{0\}, & x_1 \neq 0 \text{ and } x_2 \neq 0, \\ \emptyset, & x_1 \neq 0 \text{ and } x_2 = 0, \\ \emptyset, & x_1 = 0 \text{ and } x_2 \neq 0, \\ \{0\}, & x_1 = 0 \text{ and } x_2 = 0. \end{cases}$$

From this equation and (38), we conclude that the function f is constant along the solutions of the discontinuous dynamical system. Indeed, the level sets of the function f are exactly the diamond figures described by the solutions of the system.

Stability results

The above discussion on monotonicity is the stepping stone to provide stability results using locally Lipschitz functions and generalized gradient information. Proposition 9 provides a criterion to determine the monotonic behavior of the solutions of discontinuous dynamics along locally Lipschitz functions. This result, together with the right "positive definite" assumptions on the candidate Lyapunov function allows us to synthesize checkable stability tests. We start by formulating the natural extension of Lyapunov stability theorem for ODEs. In this and in forthcoming statements, it is convenient to adopt the convention $\max \emptyset = -\infty$.

Theorem 1. *Let $\mathcal{F} : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ be a set-valued map satisfying the hypothesis of Proposition S1. Let x_* be an equilibrium of the differential inclusion (36), and let $\mathcal{D} \subset \mathbb{R}^d$ be a domain with $x_* \in \mathcal{D}$. Let $f : \mathbb{R}^d \to \mathbb{R}$ such that*

  (i) *f is locally Lipschitz and regular on $\mathcal{D}$;*

 (ii) *$f(x_*) = 0$, and $f(x) > 0$ for $x \in \mathcal{D} \setminus \{x_*\}$;*

(iii) *$\max \widetilde{\mathcal{L}}_{\mathcal{F}} f(x) \leq 0$ for all $x \in \mathcal{D}$.*

Then, x_* is a strongly stable equilibrium of (36). In addition, if (iii) above is substituted by

(iii)' $\max \widetilde{\mathcal{L}}_{\mathcal{F}} f(x) < 0$ for all $x \in \mathcal{D} \setminus \{x_*\}$,

then x_* is a strongly asymptotically stable equilibrium of (36).

Let us apply this result to the nonsmooth harmonic oscillator. The function $(x_1, x_2) \to |x_1| + |x_2|$ verifies hypothesis *(i)-(iii)* of Theorem 1 on $\mathcal{D} = \mathbb{R}^d$. Therefore, we conclude that 0 is a strongly stable equilibrium. From the phase portrait in Figure 3(a), it is clear that 0 is not strongly asymptotically stable. The reader is invited to use Theorem 1 to deduce that the nonsmooth harmonic oscillator under dissipation, with vector field $(x_1, x_2) \mapsto (\text{sign}(x_2), -\text{sign}(x_1) - \frac{1}{2}\text{sign}(x_2))$, has 0 as a strongly asymptotically stable equilibrium.

Another important result in the theory of differential equations is the LaSalle Invariance Principle. In many situations, this principle allows us to figure out the asymptotic convergence properties of the solutions of a differential equation. Here, we build on our previous discussion to present a generalization to differential inclusions (36) and nonsmooth Lyapunov functions. Needless to say, this principle is also suitable for discontinuous differential equations. The formulation is taken from [25], and slightly generalizes the one presented in [40].

Theorem 2. *Let $\mathcal{F} : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ be a set-valued map satisfying the hypothesis of Proposition S1, and let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz and regular function. Let $S \subset \mathbb{R}^d$ be compact and strongly invariant for (36), and assume that $\max \widetilde{\mathcal{L}}_{\mathcal{F}} f(x) \leq 0$ for all $x \in S$. Then, any solution $\gamma : [t_0, \infty) \to \mathbb{R}^d$ of (36) starting at S converges to the largest weakly invariant set M contained in*

$$S \cap \overline{\{x \in \mathbb{R}^d \mid 0 \in \widetilde{\mathcal{L}}_{\mathcal{F}} f(x)\}}.$$

Moreover, if the set M is a finite collection of points, then the limit of all solutions starting at S exists and equals one of them.

Let us show next an application of this result to nonsmooth gradient flows.

Nonsmooth gradient flows revisited

Consider the nonsmooth gradient flow (29) of a locally Lipschitz function f. Assume further that the function f is regular. Let us examine how the function evolves along the solutions of the flow using the set-valued Lie derivative. Given $x \in \mathbb{R}^d$, let $a \in \widetilde{\mathcal{L}}_{-\text{Ln}(\partial f)} f(x)$. By definition, there exists $v \in F[-\text{Ln}(\partial f)](x) = -\partial f(x)$ such that

$$a = \zeta^T v, \quad \text{for all } \zeta \in \partial f(x).$$

Since the equality holds for any element in the generalized gradient of f at x, we may choose in particular $\zeta = -v \in \partial f(x)$. Therefore,

$$a = (-v)^T v = -\|v\|_2^2 \leq 0.$$

From this equation, we conclude that the elements of $\widetilde{\mathcal{L}}_{-\operatorname{Ln}(\partial f)}f$ all belong to $(-\infty, 0]$, and therefore, from equation (38), the function f monotonically decreases along the solutions of its nonsmooth gradient flow.

The application of the Lyapunov stability theorem and the LaSalle Invariance Principle above gives now rise to the following nice nonsmooth counterpart of the classical smooth results [42] for gradient flows.

Stability of nonsmooth gradient flows. Let f be a locally Lipschitz and regular function. Then, the strict minima of f are strongly stable equilibria of the nonsmooth gradient flow of f. Furthermore, if the level sets of f are bounded, then the solutions of the nonsmooth gradient flow asymptotically converge to the set of critical points of f.

As an illustration, consider the nonsmooth gradient flow of $-\operatorname{sm}_Q$ (the minimum distance to polygonal boundary function). Uniqueness of solutions for this flow can be guaranteed via Proposition 6. Regarding convergence, the application of the above result on the stability of nonsmooth gradient flows guarantee that solutions converge asymptotically to the incenter set. Indeed, one can show [36] that the incenter set is attained in finite time, and hence convergence occurs to individual points. In all, one can interpret the nonsmooth gradient flow as a "sphere-packing algorithm," in the sense that, starting from any initial point, it monotonically maximizes the radius of the largest disk contained in the polygon (that is, sm_Q!) until it reaches an incenter point. An illustration of this fact is shown in Figure 8.
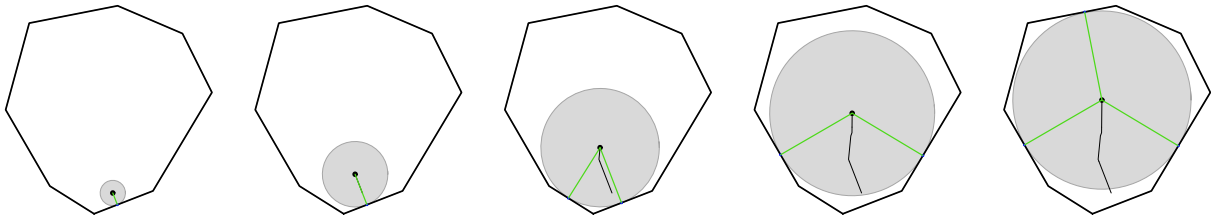

Figure 8. From left to right, evolution of the nonsmooth gradient flow of the function $-\operatorname{sm}_Q$ in a convex polygon. At each snapshot, the value of sm_Q is the radius of the largest disk (plotted in light gray) contained in the polygon with center at the current location. The flow converges in finite time to the incenter set, that for this polygon, is a singleton.

What if, instead, one is interested in packing more than one sphere within the polygon, say for example n spheres? It turns out that the "move-away-from-closest-neighbor" interaction law is a discontinuous dynamical system that solves this problem, where the solutions are understood in the Filippov sense. The interested reader is referred to [36] for various discontinuous dynamical systems that solve this and other exciting geometric optimization problems.

Finite-time convergent gradient flows of smooth functions

General results on finite-time convergence for discontinuous dynamical systems can be found in [28, 43]. Here, we briefly discuss the finite-convergence properties of a class of nonsmooth gradient flows.

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuously differentiable function, with bounded level sets. As we have mentioned before, the solutions of the gradient flow $\dot{x}(t) = -\nabla f(x(t))$ converge asymptotically toward the set of critical points of f. However, they cannot reach them *in finite time*. Here, we slightly modify the gradient flow to turn it into two different nonsmooth flows that achieve finite-time convergence.

Consider the discontinuous differential equations

$$\dot{x}(t) = -\frac{\nabla f(x(t))}{\|\nabla f(x(t))\|_2}, \tag{39}$$

$$\dot{x}(t) = -\operatorname{sign}(\nabla f(x(t))), \tag{40}$$

where $\| \cdot \|_2$ denotes the Euclidean distance and $\operatorname{sign}(x) = (\operatorname{sign}(x_1), \ldots, \operatorname{sign}(x_d)) \in \mathbb{R}^d$. We understand the solutions of these systems in the Filippov sense. The nonsmooth vector field (39) always moves in the direction of the gradient with unit speed. The nonsmooth vector field (40), instead, specifies the direction of motion via a binary quantization of the direction of the gradient. For these discontinuous systems, one can establish the following result.

Finite-time convergence of nonsmooth gradient flows. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a twice continuously differentiable function. Let $S \subset \mathbb{R}^d$ be compact and strongly invariant for (39) (resp., for (40)). If the Hessian of f is positive definite at each critical point of f in S, then each solution of (39) (resp. (40)) starting from S converges in finite time to a minimum of f.

The proof of this result builds on the stability tools presented in this section. Specifically, the LaSalle Invariance Principle can be used to establish convergence toward the set of critical points of the function. To establish finite-time convergence, one derives bounds on the evolution of the function along the solutions of the discontinuous dynamics using the set-valued Lie derivative. This analysis also allows to provide upper bounds on the convergence time. The interested reader is referred to [43] for a more comprehensive exposition of results that guarantee finite-time convergence of general discontinuous dynamics.

Finite-time consensus

Arguably, the ability to reach consensus, or agreement, upon some (a priori unknown) quantity is critical for any multi-agent system. Network coordination problems require individual agents to agree on the identity of a leader, jointly synchronize their operation, decide which specific pattern to form, balance the computational load or fuse consistently the information gathered on some spatial process. Here, we briefly comment on two discontinuous algorithms that achieve consensus in finite time, following [43].

Consider a network of n agents with states $p_1, \ldots, p_n \in \mathbb{R}$. Let $G = (\{1, \ldots, n\}, E)$ be an undirected graph with n vertices, describing the topology of the network. Two agents p_i and p_j *agree* if and only if $p_i = p_j$. The *disagreement function* $\Phi_G : \mathbb{R}^n \to [0, \infty)$ quantifies the group disagreement

$$\Phi_G(p_1, \ldots, p_n) = \frac{1}{2} \sum_{(i,j) \in E} (p_j - p_i)^2.$$

It is known [44] that, if the graph is connected, the gradient flow of Φ_G achieves consensus with an exponential rate of convergence. Actually, agents agree on the average value of their initial states –this is called *average consensus*. Regarding the nonsmooth gradient flows (39) and (40) of Φ_G, if G is connected, the first one achieves average consensus in finite time, and the second one achieves consensus on the average of the maximum and the minimum of the initial states in finite time, see [43].

Stability analysis via proximal subdifferentials of nonsmooth Lyapunov functions

This section presents stability tools for differential inclusions using lower semicontinuous functions as candidate Lyapunov functions. We make use of proximal subdifferentials to study the monotonic evolution of the candidate Lyapunov functions along the solutions of the differential inclusions. As in the previous section, we have chosen to present a few representative and useful results. We refer the interested reader to [13, 45] for a more detailed exposition.

Lie derivatives and monotonicity

Let $\mathcal{D} \subset \mathbb{R}^d$ be a domain. A lower semicontinuous function $f : \mathbb{R}^d \to \mathbb{R}$ is *weakly nonincreasing on* $\mathcal{D}$ for a set-valued map $\mathcal{F} : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ if for any $x \in \mathcal{D}$, there exists a solution $\gamma : [t_0, t_1] \to \mathbb{R}^d$ of the differential inclusion (36) starting at x and lying in $\mathcal{D}$ that satisfies

$$f(\gamma(t)) \le f(\gamma(0)) = f(x) \quad \text{for all } t \in [t_0, t_1].$$

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If in addition, f is continuous, then being weakly nonincreasing is equivalent to the property of having a solution starting at x such that $t \mapsto f(\gamma(t))$ is monotonically nonincreasing on $[t_0, t_1]$.

Similarly, a lower semicontinuous function $f : \mathbb{R}^d \to \mathbb{R}$ is *strongly nonincreasing on* $\mathcal{D}$ for a set-valued map $\mathcal{F} : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ if for any $x \in \mathcal{D}$, all solutions $\gamma : [t_0, t_1] \to \mathbb{R}^d$ of the differential inclusion (36) starting at x and lying in $\mathcal{D}$ satisfy

$$f(\gamma(t)) \leq f(\gamma(0)) = f(x) \quad \text{for all } t \in [t_0, t_1].$$

Note that being strongly nonincreasing is equivalent to the property of having $t \mapsto f(\gamma(t))$ be monotonically nonincreasing on $[t_0, t_1]$ for all solutions γ of the differential inclusion.

Given a set-valued map $\mathcal{F} : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ taking nonempty, compact values, and a lower semicontinuous function $f : \mathbb{R}^d \to \mathbb{R}$, the *lower and upper set-valued Lie derivatives* $\underline{\mathcal{L}}_{\mathcal{F}} f, \overline{\mathcal{L}}_{\mathcal{F}} f : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R})$ of f with respect to $\mathcal{F}$ at x are defined by, respectively

$$\underline{\mathcal{L}}_{\mathcal{F}} f(x) = \{ a \in \mathbb{R} \mid \text{there exists } \zeta \in \partial_P f(x) \text{ such that } a = \min\{\zeta^T v \mid v \in \mathcal{F}(x)\}\},$$
$$\overline{\mathcal{L}}_{\mathcal{F}} f(x) = \{ a \in \mathbb{R} \mid \text{there exists } \zeta \in \partial_P f(x) \text{ such that } a = \max\{\zeta^T v \mid v \in \mathcal{F}(x)\}\},$$

If, in addition, $\mathcal{F}$ takes convex values, then for each $\zeta \in \partial_P f(x)$, the set $\{\zeta^T v \mid v \in \mathcal{F}(x)\}$ is a closed interval of the form $[\min\{\zeta^T v \mid v \in \mathcal{F}(x)\}, \max\{\zeta^T v \mid v \in \mathcal{F}(x)\}]$. Note that the lower and upper set-valued Lie derivatives at a point x might be empty.

The lower and upper set-valued Lie derivatives play a similar role for lower semicontinuous functions to the one played by the set-valued Lie derivative $\widetilde{\mathcal{L}}_{\mathcal{F}} f$ for locally Lipschitz functions. These objects allow us to study how the function f evolves along the solutions of a differential inclusion without having to obtain them in closed form. Specifically, we have the following result. In this and in forthcoming statements, it is convenient to adopt the convention $\sup \emptyset = -\infty$.

Proposition 10. *Let $\mathcal{F} : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ be a set-valued map satisfying the hypothesis of Proposition S1, and consider the associated differential inclusion* (36). *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a lower semicontinuous function, and $\mathcal{D} \subset \mathbb{R}^d$ open. Then,*

(i) *The function f is weakly nonincreasing on $\mathcal{D}$ if and only if*

$$\sup \underline{\mathcal{L}}_{\mathcal{F}} f(x) \leq 0, \quad \text{for all } x \in \mathcal{D};$$

(ii) *If, in addition, either $\mathcal{F}$ is locally Lipschitz on $\mathcal{D}$, or $\mathcal{F}$ is continuous on $\mathcal{D}$ and f is locally Lipschitz on $\mathcal{D}$, then f is strongly nonincreasing on $\mathcal{D}$ if and only if*

$$\sup \overline{\mathcal{L}}_{\mathcal{F}} f(x) \leq 0, \quad \text{for all } x \in \mathcal{D}.$$

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Let us illustrate this result in a particular example.

Cart on a circle

Consider, following [45, 46], the driftless control system on $\mathbb{R}^2$

$$\dot{x}_1 = (x_1^2 - x_2^2)u,$$
$$\dot{x}_2 = 2x_1 x_2 u,$$

with $u \in \mathbb{R}$. The phase portrait of the vector field $(x_1, x_2) \mapsto g(x_1, x_2) = (x_1^2 - x_2^2, 2x_1 x_2)$ is plotted in Figure 9(a).
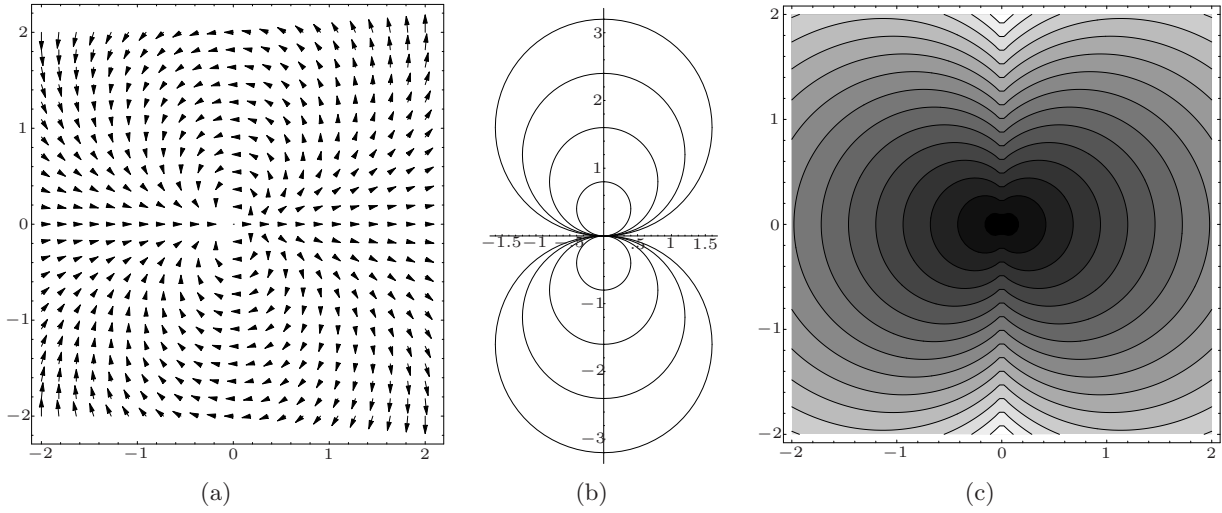

(a)   (b)   (c)

Figure 9. Cart on a circle. The plot in (a) shows the phase portrait of the vector field $(x_1, x_2) \mapsto (x_1^2 - x_2^2, 2x_1 x_2)$, the plot in (b) shows its integral curves, and the plot in (c) shows the contour plot of the function $0 \neq (x_1, x_2) \mapsto \frac{x_1^2 + x_2^2}{\sqrt{x_1^2 + x_2^2 + |x_1|}}$, $(0,0) \mapsto 0$.

Alternatively, consider the associated set-valued map $\mathcal{F} : \mathbb{R}^2 \to \mathfrak{B}(\mathbb{R}^2)$ defined by $\mathcal{F}(x_1, x_2) = \{g(x_1, x_2)\, u \mid u \in \mathbb{R}\}$. Note that $\mathcal{F}$ does not take compact values. Therefore, instead of considering $\mathcal{F}$, we take any nondecreasing map $\sigma : [0, \infty) \to [0, \infty)$, and define the set-valued map $\mathcal{F}_\sigma : \mathbb{R}^2 \to \mathfrak{B}(\mathbb{R}^2)$ given by $\mathcal{F}_\sigma(x_1, x_2) = \{g(x_1, x_2)\, u \in \mathbb{R}^2 \mid |u| \leq \sigma(\|(x_1, x_2)\|_2)\}$.

Consider the locally Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x_1, x_2) = \begin{cases} \frac{x_1^2 + x_2^2}{\sqrt{x_1^2 + x_2^2 + |x_1|}}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

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The level set curves of this function are depicted in Figure 9(b). Let us determine how f evolves along the solutions of the control system by using the lower and upper set-valued Lie derivatives. First, let us compute the proximal subdifferential of f. Using the fact that f is twice continuously differentiable on the open right and left half-planes, together with the geometric interpretation of proximal subgradients, we obtain

$$\partial_P f(x_1, x_2) = \begin{cases} \left\{ \left(-\frac{x_1^2+x_2^2-2x_1\sqrt{x_1^2+x_2^2}}{x_1^2+x_2^2+x_1\sqrt{x_1^2+x_2^2}}, \frac{x_2(2x_1+\sqrt{x_1^2+x_2^2})}{(x_1+\sqrt{x_1^2+x_2^2})^2} \right) \right\}, & x_1 > 0, \\ \emptyset, & x_1 = 0, \\ \left\{ \left(\frac{x_1^2+x_2^2+2x_1\sqrt{x_1^2+x_2^2}}{x_1^2+x_2^2-x_1\sqrt{x_1^2+x_2^2}}, \frac{x_2(-2x_1+\sqrt{x_1^2+x_2^2})}{(x_1-\sqrt{x_1^2+x_2^2})^2} \right) \right\}, & x_1 < 0. \end{cases}$$

With this information, we compute the set

$$\{ \zeta^T v \mid \zeta \in \partial_P f(x_1, x_2), \ v \in \mathcal{F}_\sigma(x_1, x_2) \} = \begin{cases} \left\{ u \frac{(x_1^2+x_2^2)^2}{x_1^2+x_2^2+x_1\sqrt{x_1^2+x_2^2}} \mid |u| \leq \sigma(\|(x_1, x_2)\|_2) \right\}, & x_1 > 0, \\ \emptyset, & x_1 = 0, \\ \left\{ -u \frac{(x_1^2+x_2^2)^2}{x_1^2+x_2^2-x_1\sqrt{x_1^2+x_2^2}} \mid |u| \leq \sigma(\|(x_1, x_2)\|_2) \right\}, & x_1 < 0. \end{cases}$$

We are now ready to compute the lower and upper set-valued Lie derivatives as

$$\underline{\mathcal{L}}_{\mathcal{F}} f(x_1, x_2) = \begin{cases} -\sigma(\|(x_1, x_2)\|_2) \frac{(x_1^2+x_2^2)^{3/2}}{\sqrt{x_1^2+x_2^2}+|x_1|}, & x_1 \neq 0, \\ -\infty, & x_1 = 0, \end{cases}$$

$$\overline{\mathcal{L}}_{\mathcal{F}} f(x_1, x_2) = \begin{cases} \sigma(\|(x_1, x_2)\|_2) \frac{(x_1^2+x_2^2)^{3/2}}{\sqrt{x_1^2+x_2^2}+|x_1|}, & x_1 \neq 0, \\ -\infty, & x_1 = 0. \end{cases}$$

Therefore $\sup \underline{\mathcal{L}}_{\mathcal{F}} f(x_1, x_2) \leq 0$, for all $(x_1, x_2) \in \mathbb{R}^2$. Using now Proposition 10(i), we deduce that the function f is weakly nonincreasing on $\mathbb{R}^2$. Since f is continuous, this fact is equivalent to saying that there exists a choice of control input u such that the solution γ of the resulting dynamical system satisfies that $t \mapsto f(\gamma(t))$ is monotonically nonincreasing.

Stability results

The results presented in the previous section establishing the monotonic behavior of lower semicontinuous functions allow us to provide tools for stability analysis. We present here an exposition parallel to the one for locally Lipschitz functions and generalized gradients. We start by presenting a result on Lyapunov stability.

Theorem 3. *Let $\mathcal{F} : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ be a set-valued map satisfying the hypothesis of Proposition S1. Let x_* be an equilibrium of the differential inclusion (36), and let $\mathcal{D} \subset \mathbb{R}^d$ be a domain with $x_* \in \mathcal{D}$. Let $f : \mathbb{R}^d \to \mathbb{R}$ and assume that*

(i) $\mathcal{F}$ is continuous on $\mathcal{D}$ and f is locally Lipschitz on $\mathcal{D}$, or $\mathcal{F}$ is locally Lipschitz on $\mathcal{D}$ and f is lower semicontinuous on $\mathcal{D}$, and f is continuous at x_*;

(ii) $f(x_*) = 0$, and $f(x) > 0$ for $x \in \mathcal{D} \setminus \{x_*\}$;

(iii) $\sup \overline{\mathcal{L}}_{\mathcal{F}} f(x) \leq 0$ for all $x \in \mathcal{D}$.

Then, x_* is a strongly stable equilibrium of (36). In addition, if (iii) above is substituted by

(iii)' $\sup \overline{\mathcal{L}}_{\mathcal{F}} f(x) < 0$ for all $x \in \mathcal{D} \setminus \{x_*\}$,

then x_* is a strongly asymptotically stable equilibrium of (36).

A similar result can be stated for weakly stable equilibria substituting (i) by "(i') f is continuous on $\mathcal{D}$," and the upper set-valued Lie derivative by the lower set-valued Lie derivative in (iii) and (iii'). Note that, if the differential inclusion (36) has unique solutions starting from any initial condition, then the notions of strong and weak stability coincide, and it is sufficient to verify the simpler requirements of the result for weak stability.

In a similar way to the case of continuous differential equations, global asymptotic stability can be established by requiring the Lyapunov function f to be continuous and radially unbounded. Indeed, this type of global results are commonly invoked when dealing with the stabilization of control systems by referring to control Lyapunov functions [45] or Lyapunov pairs [13]. Two lower semicontinuous functions $f, g : \mathbb{R}^d \to \mathbb{R}$ are a *Lyapunov pair* for an equilibrium $x_* \in \mathbb{R}^d$ if they satisfy that $f(x), g(x) \geq 0$ for $x \in \mathbb{R}^d$, and $g(x) = 0$ if and only if $x = x_*$; f is radially unbounded, and moreover,

$$\sup \underline{\mathcal{L}}_{\mathcal{F}} f(x) \leq -g(x), \quad \text{for all } x \in \mathbb{R}^d.$$

If an equilibrium x_* of (36) admits a Lyapunov pair, then one can show that there exists at least one solution starting from any initial condition that asymptotically converges to the equilibrium, see [13].

As an application of this discussion and the version of Theorem 3 for weak stability, consider the cart on a circle example. Setting $x_* = (0, 0)$ and $\mathcal{D} = \mathbb{R}^2$, and taking into account our previous computation of the lower set-valued Lie derivative, we conclude that $(0, 0)$ is a (globally) weakly asymptotically stable equilibrium.

We now turn our attention to the extension of LaSalle Invariance Principle for differential inclusions using lower semicontinuous functions and proximal subdifferentials.

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Theorem 4. *Let $\mathcal{F} : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ be a set-valued map satisfying the hypothesis of Proposition S1, and let $f : \mathbb{R}^d \to \mathbb{R}$. Assume either $\mathcal{F}$ is continuous and f is locally Lipschitz, or $\mathcal{F}$ is locally Lipschitz and f is continuous. Let $S \subset \mathbb{R}^d$ be compact and strongly invariant for (36), and assume that $\sup \overline{\mathcal{L}}_\mathcal{F} f(x) \le 0$ for all $x \in S$. Then, any solution $\gamma : [t_0, \infty) \to \mathbb{R}^d$ of (36) starting at S converges to the largest weakly invariant set M contained in*

$$S \cap \overline{\{x \in \mathbb{R}^d \mid 0 \in \overline{\mathcal{L}}_\mathcal{F} f(x)\}}$$

Moreover, if the set M is a finite collection of points, then the limit of all solutions starting at S exists and equals one of them.

Let us apply this result to gradient differential inclusions.

Gradient differential inclusions revisited

Consider the gradient differential inclusion (35) associated to a continuous and convex function $f : \mathbb{R}^d \to \mathbb{R}$. Let us study here the asymptotic behavior of the solutions. From our previous discussion, we know that solutions exist and are unique. In particular, this fact means that in this case the notions of weakly nonincreasing and strongly nonincreasing function coincide. Therefore, let us simply show that the function f is weakly nonincreasing on $\mathbb{R}^d$ for the gradient differential inclusion.

For any $\zeta \in \partial_P f(x)$, there is $v = -\zeta \in -\partial_P f(x)$ such that $\zeta^T v = -\|\zeta\|_2^2 \le 0$. In particular, this implies

$$\underline{\mathcal{L}}_{-\partial_P f} f(x) \le 0, \quad \text{for all } x \in \mathbb{R}^d.$$

Proposition 10(i) now guarantees that f is weakly nonincreasing on $\mathbb{R}^d$. Since the solutions of the gradient differential inclusion are unique, f is monotonically nonincreasing.

The application of the Lyapunov stability theorem and the LaSalle Invariance Principle above gives now rise to the following nice nonsmooth counterpart of the classical smooth results for gradient flows.

Stability of gradient differential inclusions. Let f be a continuous and convex function. Then, the strict minima of f are strongly stable equilibria of the gradient differential inclusion associated to f. Furthermore, if the level sets of f are bounded, then the solutions of the gradient differential inclusion asymptotically converge to the set of minima of f.

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Stabilization of control systems

Consider an autonomous control system on $\mathbb{R}^d$ of the form

$$\dot{x} = X(x, u), \tag{41}$$

where $X : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ (note that the space of admissible controls is $\mathcal{U} \subset \mathbb{R}^m$). The system is locally (respectively globally) *continuously stabilizable* if there exists a continuous map $k : \mathbb{R}^d \to \mathbb{R}^m$ such that the closed-loop system

$$\dot{x} = X(x, k(x))$$

is locally (respectively globally) asymptotically stable at the origin. The celebrated result by Brockett [1], see also [2, 3], states that many control systems are not continuously stabilizable.

Theorem 5. *Let* $X : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ *be continuous and* $X(0,0) = 0$. *A necessary condition for the existence of a continuous stabilizer of the control system* (41) *is that* X *maps any neighborhood of the origin in* $\mathbb{R}^d \times \mathbb{R}^m$ *onto some neighborhood of the origin in* $\mathbb{R}^d$.

In particular, Theorem 5 implies that driftless control systems of the form

$$\dot{x} = u_1 X_1(x) + \cdots + u_m X_m(x), \tag{42}$$

with $m < n$, and $X_i : \mathbb{R}^d \to \mathbb{R}^d$, $i \in \{1, \ldots, m\}$ continuous, cannot be stabilized by a continuous feedback.

The condition in Theorem 5 is only necessary. There exist control systems that satisfy it, and still cannot be stabilized by means of a continuous stabilizer. The cart on a circle example is one of them. The map $((x_1, x_2), u) \to g(x)u$ is onto any neighborhood of $(0,0)$. However, it cannot be stabilized with a continuous $k : \mathbb{R}^2 \to \mathbb{R}$, see [45] for various ways to justify it.

The obstruction to the existence of continuous stabilizers has motivated the search for time-varying and discontinuous feedback stabilizers. Regarding the latter, an immediate question pops up: if one uses a discontinuous map $k : \mathbb{R}^m \to \mathbb{R}^d$, how should the solutions of the resulting discontinuous dynamical system $\dot{x} = X(x, k(x))$ be understood? From the previous discussion, we know that Caratheodory solutions are not a good candidate, since in many situations they fail to exist. The following result [47, 48], shows that Filippov solutions are not a good candidate either.

Theorem 6. *Let* $X : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ *be continuous and* $X(0,0) = 0$. *Assume that for each* $U \subset \mathbb{R}^m$ *and each* $x \in \mathbb{R}^d$, *one has* $X(x, \mathrm{co}\, U) = \mathrm{co}\, X(x, U)$. *Then, a necessary condition for the existence of a measurable, locally bounded stabilizer of the control system* (41) *(where solutions are understood in the Filippov sense) is that* X *maps any neighborhood of the origin in* $\mathbb{R}^d \times \mathbb{R}^m$ *onto some neighborhood of the origin in* $\mathbb{R}^d$.

In particular, driftless control systems of the form (42) cannot be stabilized by means of a discontinuous feedback if solutions are understood in the Filippov sense. This impossibility result, however, can be overcome if solutions are understood in the sample-and-hold sense, as shown in [32]. This work used this notion to solve the open question concerning the relationship between asymptotic controllability and feedback stabilization.

Let us briefly discuss this result in the light of our previous exposition. Consider the differential inclusion (18) associated with the control system (41). The system (41) is *(open loop) globally asymptotically controllable* (to the origin) if 0 is a Lyapunov stable equilibrium of (18), and every point $x \in \mathbb{R}^d$ has the property that there exists a solution of (18) satisfying $x(0) = x$ and $\lim_{t\to\infty} x(t) = 0$. On the other hand, a feedback $k : \mathbb{R}^d \to \mathbb{R}^m$ *stabilizes* the system (41) *in the sample-and-hold sense* if, for all $x_0 \in \mathbb{R}^d$ and all $\varepsilon \in (0, \infty)$, there exist $\delta, T \in (0, \infty)$ such that, for any partition π of $[0, t_1]$ with $\text{diam}(\pi) < \delta$, the corresponding π-solution γ of (41) starting at x_0 satisfies $\|\gamma(t)\|_2 \leq \varepsilon$ for all $t \geq T$.

The following result states that both notions, global asymptotic controllability and the existence of a feedback stabilizer, are equivalent.

Theorem 7. *Let $X : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ be continuous and $X(0, 0) = 0$. Then, the control system (41) is globally asymptotically controllable if and only if it admits a measurable, locally bounded stabilizer in the sample-and-hold sense.*

The implication from right to left is clear. The converse implication is proved by explicit construction of the stabilizer, and is based on the fact that the control system (41) is globally asymptotically controllable if and only if it admits a continuous Lyapunov pair, see [49]. Using the continuous Lyapunov function provided by this characterization, one constructs explicitly the discontinuous feedback for the control system (41), see [32, 45]. The existence of a Lyapunov pair "in the sense of generalized gradients" (that is, when instead of using the lower set-valued Lie derivative involving proximal subdifferential, one uses the set-valued Lie derivative involving the generalized gradient) turns out to be equivalent to the existence of a stabilizing feedback in the sense of Filippov, see [50].

As an illustration, consider the cart on a circle example. We have already shown that $(0, 0)$ is a globally weakly asymptotically stable equilibrium of the differential inclusion associated with the control system. Therefore, the control system is globally asymptotically controllable, and can be stabilized in the sample-and-hold sense by means of a discontinuous feedback. The stabilizing feedback that results from the proof of Theorem 7 is the following, see [45, 51]: if to the left of the x_2 axis, move in the direction of the vector field g, if to the right of the x_2 axis, move in the opposite direction of the vector field g, and make an arbitrary decision on the x_2-axis. The stabilizing nature of this feedback can be graphically checked in Figure 9(a) and (b).

Remarkably, for systems affine in the control, there exist [52] stabilizing feedbacks whose discontinuities form a set of measure zero, and, moreover, the discontinuity set is repulsive for the solutions of the closed-loop system. In particular, this fact means that in applying the feedback, the solutions can be understood in the Caratheodory sense. This situation is exactly what we see in the cart on a circle example.

Conclusions

We have presented an introductory tutorial on discontinuous dynamical systems. We have begun by reviewing the classical notion of solution for ordinary differential equations. We have illustrated in various examples the pertinence of the continuity and Lipschitzness hypotheses that guarantee the existence and uniqueness of classical solutions. Our discussion has motivated the need for more general notions than the classical one. From this point, three main themes have guided our discussion: appropriate notions of solution for discontinuous systems, nonsmooth analysis and gradient information of candidate Lyapunov functions, and nonsmooth stability tools to characterize the asymptotic behavior of solutions.

Regarding the first theme, we have introduced the notions of Caratheodory, Filippov and sample-and-hold solutions, discussed existence and uniqueness results, and examined various examples to illustrate them. Regarding the second theme, we have presented two sets of alternative tools: on the one hand, locally Lipschitz functions and their generalized gradients, and on the other hand, lower semicontinuous functions and their proximal subdifferentials. We have provided tools for the explicit computation of these gradient notions, and discussed suitable generalizations of the concept of critical points and directions of descent. As a paradigmatic example, we have paid special attention to the gradient flow of both locally Lipschitz and lower semicontinuous functions. Finally, regarding the third theme, we have introduced Lie derivative tools to analyze the monotonic behavior of candidate Lyapunov functions. Making use of these tools, we have presented generalizations of the Lyapunov stability theorem and the LaSalle Invariance Principle for discontinuous systems. We have illustrated the application of these results with the class of nonsmooth gradient flows and other examples. For reference, the sidebar "Index of Symbols" presents the symbols corresponding to the main mathematical concepts used throughout the article.

Numerous important issues have been left out. The topic of discontinuous dynamical systems is a vast one, and we have focused our attention on the above-mentioned themes with the aim of providing a coherent exposition. We hope that this tutorial serves as a guided motivation for the reader to further explore the exciting topic of discontinuous systems. The list of references of this manuscript provides a good starting point to undertake this endeavor.

Acknowledgments

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Sidebar 1: Locally Lipschitz Functions

A function $f : \mathbb{R}^d \to \mathbb{R}^m$ is *locally Lipschitz at* $x \in \mathbb{R}^d$ if if there exist $L_x, \varepsilon \in (0, \infty)$ such that

$$\|f(y) - f(y')\|_2 \leq L_x \|y - y'\|_2,$$

for all $y, y' \in B(x, \varepsilon)$. A locally Lipschitz function at x is continuous at x, but the converse is not true ($f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt{|x|}$, is continuous at 0, but not locally Lipschitz at 0). A function is *locally Lipschitz on* $S \subset \mathbb{R}^d$ if it is locally Lipschitz at x, for all $x \in S$. We abbreviate "f is locally Lipschitz on $\mathbb{R}^d$" by simply saying "f is locally Lipschitz." Note that continuously differentiable functions at x are locally Lipschitz at x, but the converse is not true ($f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$, is locally Lipschitz at 0, but not differentiable at 0). Here, functions like $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^m$, that depend explicitly on time, are *locally Lipschitz at* $x \in \mathbb{R}^d$ if there exists $\varepsilon \in (0, \infty)$ and $L_X : \mathbb{R} \to (0, \infty)$ such that $\|f(t, y) - f(t, y')\|_2 \leq L_x(t) \|y - y'\|_2$, for all $t \in \mathbb{R}$ and $y, y' \in B(x, \varepsilon)$.

Sidebar 2: Absolutely continuous functions

A function $\gamma : [a, b] \to \mathbb{R}$ is *absolutely continuous* if for all $\varepsilon \in (0, \infty)$, there exists $\delta \in (0, \infty)$ such that any finite collection $(a_1, b_1), \ldots, (a_n, b_n)$ of disjoint open intervals contained in $[a, b]$ with $\sum_{i=1}^{n} (b_i - a_i) < \delta$ verifies

$$\sum_{i=1}^{n} |\gamma(b_i) - \gamma(a_i)| < \varepsilon.$$

Locally Lipschitz functions are absolutely continuous. The function $\gamma : [0, 1] \to \mathbb{R}$, $\gamma(x) = \sqrt{x}$, is absolutely continuous but not locally Lipschitz at 0. Absolutely continuous functions are (uniformly) continuous. The function $\gamma : [-1, 1] \to \mathbb{R}$ defined by $\gamma(t) = t \sin\left(\frac{1}{t}\right)$ for $t \neq 0$ and $\gamma(0) = 0$ is continuous, but not absolutely continuous. Finally, absolutely continuous functions are differentiable almost everywhere.

Sidebar 3: Set-valued Maps

A set-valued map, as its name suggests, are maps that have sets as images. More formally, let $\mathfrak{B}(S)$ be the collection of all possible subsets of $S \subset \mathbb{R}^d$. We consider (non-autonomous) set-valued maps of the form $\mathcal{F} : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathfrak{B}(\mathbb{R}^d)$. The map $\mathcal{F}$ assigns each point $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ to the set $\mathcal{F}(t, x) \subset \mathbb{R}^d$. One can develop a complete analysis for set-valued maps, very much like in the case of standard regular maps, see, for instance [35]. Here, we are mainly interested in concepts related to boundedness and continuity, that we define next for completeness.

A set-valued map $\mathcal{F} : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathfrak{B}(\mathbb{R}^d)$ is *locally bounded* (respectively *locally essentially bounded*) at $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ if there exist $\varepsilon \in (0, \infty)$ and an integrable function $m : [t, t + \delta] \rightarrow (0, \infty)$ such that $\|z\|_2 \leq m(s)$ for all $z \in \mathcal{F}(s, y)$, all $s \in [t, t + \delta]$, and all $y \in B(x, \varepsilon)$ (respectively, almost all $y \in B(x, \varepsilon)$ in the sense of Lebesgue measure).

An (autonomous) set-valued map $\mathcal{F} : \mathbb{R}^d \rightarrow \mathfrak{B}(\mathbb{R}^d)$ is *upper semicontinuous* (respectively, *lower semicontinuous*) at $x \in \mathbb{R}^d$ if for all $\varepsilon \in (0, \infty)$, there exists $\delta \in (0, \infty)$ such that $\mathcal{F}(y) \subset \mathcal{F}(x) + B(0, \varepsilon)$ (respectively, $\mathcal{F}(x) \subset \mathcal{F}(y) + B(0, \varepsilon)$) for all $y \in B(x, \delta)$. A set-valued map $\mathcal{F} : \mathbb{R}^d \rightarrow \mathfrak{B}(\mathbb{R}^d)$ is *continuous* at $x \in \mathbb{R}^d$ if it is both upper and lower semicontinuous at $x \in \mathbb{R}^d$. Finally, a set-valued map $\mathcal{F} : \mathbb{R}^d \rightarrow \mathfrak{B}(\mathbb{R}^d)$ is *locally Lipschitz* at $x \in \mathbb{R}^d$ if there exist $L_x, \varepsilon \in (0, \infty)$ such that

$$\mathcal{F}(y') \subset \mathcal{F}(y) + L_x \|y - y'\|_2 \overline{B}(0, 1),$$

for all $y, y' \in B(x, \varepsilon)$. A locally Lipschitz set-valued map at x is upper semicontinuous at x, but the converse is not true.

Sidebar 4: Differential Inclusions and Caratheodory Solutions

Differential inclusions are a generalization of differential equations: at each state, they specify a range of possible evolutions, rather than a single one. These objects are defined by means of set-valued maps, see the sidebar "Set-valued maps." The *differential inclusion* associated to $\mathcal{F} : \mathbb{R} \times \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ is an equation of the form

$$\dot{x}(t) \in \mathcal{F}(t, x(t)). \tag{S1}$$

A point $x_* \in \mathbb{R}^d$ is an *equilibrium* of the differential inclusion if $0 \in \mathcal{F}(t, x_*)$ for all $t \in \mathbb{R}$. We define the notion of solution of a differential inclusion *a la Caratheodory*. The flexibility provided by the differential inclusion makes things work under fairly general conditions.

A *(Caratheodory) solution of* (S1) defined on $[t_0, t_1] \subset \mathbb{R}$ is an absolutely continuous map $\gamma : [t_0, t_1] \to \mathbb{R}^d$ such that $\dot{\gamma}(t) \in \mathcal{F}(t, \gamma(t))$ for almost every $t \in [t_0, t_1]$. The existence of at least a solution starting from each initial condition is guaranteed by the following result (see, for instance, [3, 15]).

Proposition S1. *Let $\mathcal{F} : \mathbb{R} \times \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$ be locally bounded and take nonempty, compact and convex values. Assume that, for each $t \in \mathbb{R}$, the set-valued map $x \mapsto \mathcal{F}(t, x)$ is upper semicontinuous, and, for each $x \in \mathbb{R}^d$, the set-valued map $t \mapsto \mathcal{F}(t, x)$ is measurable. Then, for any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$, there exists a solution of (S1) with initial condition $x(t_0) = x_0$.*

This result is sufficient for our purposes. The reader is invited to find in the literature other existence results that work under different assumptions, see for instance [3, 13]. Uniqueness of solutions of differential inclusions is guaranteed by the following result.

Proposition S2. *Under the hypothesis of Proposition S1, further assume that for all $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, there exist $L_X(t), \varepsilon \in (0, \infty)$ such that for almost every $y, y' \in B(x, \varepsilon)$, one has*

$$(v - w)^T (y - y') \leq L_x(t) \|y - y'\|_2^2, \tag{S2}$$

for all $v \in \mathcal{F}(t, y)$ and $w \in \mathcal{F}(t, y')$. Assume that the function $t \mapsto L_X(t)$ is integrable. Then, for any $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$, there exists a unique solution of (S1) with initial condition $x(t_0) = x_0$.

Let us present an example of the application of Propositions S1 and S2. Following [35], consider the set-valued map $\mathcal{F} : \mathbb{R} \to \mathfrak{B}(\mathbb{R})$ defined by

$$\mathcal{F}(x) = \begin{cases} 0, & x \neq 0, \\ [-1, 1], & x = 0. \end{cases}$$

Note that $\mathcal{F}$ is upper semicontinuous, but not lower semicontinuous (and hence, it is not continuous). This set-valued map verifies all the hypotheses in Proposition S1, and therefore solutions exist starting from any initial condition. In addition, $\mathcal{F}$ satisfies equation (S2) as long as y and y' are different from 0. Therefore, Proposition S2 guarantees uniqueness. Actually, the solution of $\dot{x}(t) \in \mathcal{F}(x(t))$ starting from any initial condition is just the equilibrium solution.

Sidebar 5: Piecewise Continuous Vector Fields

To guarantee uniqueness of solution for a piecewise continuous vector field, we can not resort to Proposition 5. To see this, let $X : \mathbb{R}^d \to \mathbb{R}^d$, $d \geq 2$, be piecewise continuous, and consider a point of discontinuity $x \in S_X$. For simplicity, assume it belongs to the boundaries of just two domains, that is, $x \in \partial\mathcal{D}_i \cap \partial\mathcal{D}_j$ (the argument proceeds similarly for the general case). For $\varepsilon \in (0, \infty)$, let us show that equation (16) is violated on a set of non-zero measure contained in $B(x, \varepsilon)$. Notice that

$$(X(y) - X(y'))^T(y - y') = \|X(y) - X(y')\|_2 \, \|y - y'\|_2 \, \cos\alpha(y, y'),$$

where $\alpha(y, y') = \angle(X(y) - X(y'), y - y')$ is the angle between the vectors $X(y) - X(y')$ and $y - y'$. Therefore, equation (16) is equivalent to

$$\|X(y) - X(y')\|_2 \, \cos\alpha(y, y') \leq L_X \|y - y'\|_2. \tag{S3}$$

Consider the vectors $X_{|\overline{\mathcal{D}}_i}(x)$ and $X_{|\overline{\mathcal{D}}_j}(x)$. Since X is discontinuous at x, we have $X_{|\overline{\mathcal{D}}_i}(x) \neq X_{|\overline{\mathcal{D}}_j}(x)$. Take any $y \in \mathcal{D}_i \cap B(x, \varepsilon)$ and $y' \in \mathcal{D}_j \cap B(x, \varepsilon)$. Note that as y and y' tend to x, the vector $X(y) - X(y')$ tends to $X_{|\overline{\mathcal{D}}_i}(x) - X_{|\overline{\mathcal{D}}_j}(x)$. Consider then a straight line L that crosses S_X, passes through x, and forms a small angle $\beta > 0$ with $X_{|\overline{\mathcal{D}}_i}(x) - X_{|\overline{\mathcal{D}}_j}(x)$ (see Figure S1).
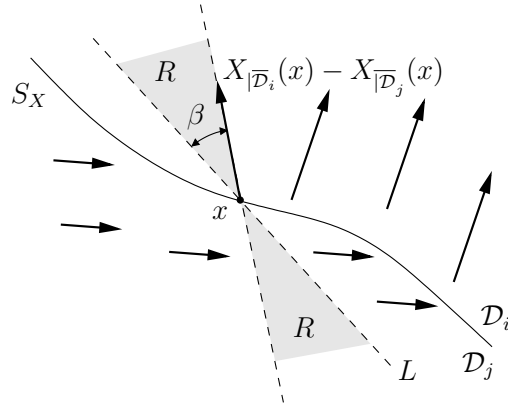
Figure S1. Piecewise continuous vector field. The vector field has a unique Filippov solution starting from any initial condition –solutions that reach S_X coming from $\mathcal{D}_j$ cross it, and then continue in $\mathcal{D}_i$. However, Proposition 5 cannot be invoked to conclude uniqueness.

Let R be the set enclosed by the line L and the line in the direction of the vector $X_{|\overline{\mathcal{D}}_i}(x) - X_{|\overline{\mathcal{D}}_j}(x)$. If $y \in \mathcal{D}_i \cap R$ and $y' \in \mathcal{D}_j \cap R$ tend to x, we deduce that $\|y - y'\|_2 \to 0$ while at the same time

$$\|X(y) - X(y')\|_2 \,|\cos\alpha(y, y')| \geq$$
$$\|X(y) - X(y')\|_2 \cos\beta \longrightarrow \|X_{|\overline{\mathcal{D}}_i}(x) - X_{|\overline{\mathcal{D}}_j}(x)\|_2 \cos\beta > 0.$$

Therefore, it cannot exist $L_X \in (0, \infty)$ such that equation (S3) is verified for $y \in R \cap \mathcal{D}_i \cap B(x, \varepsilon)$ and $y' \in R \cap \mathcal{D}_j \cap B(x, \varepsilon)$.

Sidebar 6: Regular Functions

Let us recall here the notion of regular function. To introduce it, we need to first define what right directional derivatives and generalized right directional derivatives are. Given $f : \mathbb{R}^d \to \mathbb{R}$, the *right directional derivative* of f at x in the direction of $v \in \mathbb{R}^d$ is defined as

$$f'(x, v) = \lim_{h \to 0^+} \frac{f(x + hv) - f(x)}{h},$$

when this limits exists. On the other hand, the *generalized directional derivative* of f at x in the direction of $v \in \mathbb{R}^d$ is defined as

$$f^o(x; v) = \limsup_{\substack{y \to x \\ h \to 0^+}} \frac{f(y + hv) - f(y)}{h} = \lim_{\substack{\delta \to 0^+ \\ \varepsilon \to 0^+}} \sup_{\substack{y \in B(x, \delta) \\ h \in [0, \varepsilon)}} \frac{f(y + hv) - f(y)}{h}.$$

This latter notion has the advantage of always being well-defined. In general, these directional derivatives may not be equal. When they are, we call the function regular. More formally, a function $f : \mathbb{R}^d \to \mathbb{R}$ is *regular at* $x \in \mathbb{R}^d$ if for all $v \in \mathbb{R}^d$, the right directional derivative of f at x in the direction of v exists, and $f'(x; v) = f^o(x; v)$. A continuously differentiable function at x is regular at x. Also, a convex and locally Lipschitz function at x is regular (cf. [33, Proposition 2.3.6]). An example of a non-regular function is $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -|x|$. The function is continuously differentiable everywhere except for zero, so it is regular on $\mathbb{R} \setminus \{0\}$. However, its directional derivatives

$$f'(0; v) = \begin{cases} -v, & v > 0, \\ v, & v < 0, \end{cases} \qquad f^o(0; v) = \begin{cases} v, & v > 0, \\ -v, & v < 0, \end{cases}$$

do not coincide. Hence, the function is not regular at 0.

Sidebar 7: Index of Symbols

The following is a list of the symbols used throughout the article.

Symbol	Description
$G[X]$	Set-valued map associated with a control system $X : \mathbb{R} \times \mathbb{R}^d \times \mathcal{U} \to \mathbb{R}^d$
$\mathrm{co}(S)$	Convex hull of a set $S \subset \mathbb{R}^d$
$\mathrm{diam}(\pi)$	Diameter of the partition π
$f^o(x,v)$	Generalized directional derivative of the function $f : \mathbb{R}^d \to \mathbb{R}$ at $x \in \mathbb{R}^d$ in the direction of $v \in \mathbb{R}^d$
$f'(x,v)$	Right directional derivative of the function $f : \mathbb{R}^d \to \mathbb{R}$ at $x \in \mathbb{R}^d$ in the direction of $v \in \mathbb{R}^d$
S_X	Set of points where the vector field $X : \mathbb{R}^d \to \mathbb{R}^d$ is discontinuous
$\mathrm{dist}(p,S)$	Euclidean distance from the point $p \in \mathbb{R}^d$ to the set $S \subset \mathbb{R}^d$
$F[X]$	Filippov set-valued map associated with a vector field $X : \mathbb{R}^d \to \mathbb{R}^d$
∂f	Generalized gradient of the locally Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$
∇f	Gradient of the differentiable function $f : \mathbb{R}^d \to \mathbb{R}$
$\mathrm{Ln}(S)$	Least-norm elements in the closure of the set $S \subset \mathbb{R}^d$
$\Omega(\gamma)$	Set of limit points of a curve γ
$\mathcal{N}$	Nearest-neighbor map
n_e	Unit normal to the edge e of a polygon Q pointing toward the interior of Q
Ω_f	Set of points where the locally Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ fails to be differentiable
π	Partition of a closed interval
$\mathfrak{B}(S)$	Set whose elements are all the possible subsets of $S \subset \mathbb{R}^d$
$\partial_P f$	Proximal subdifferential of the lower semicontinuous function $f : \mathbb{R}^d \to \mathbb{R}$

$\widetilde{\mathcal{L}}_{\mathcal{F}} f$	Set-valued Lie derivative of the locally Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ with respect to the set-valued map $\mathcal{F} : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$
$\widetilde{\mathcal{L}}_X f$	Set-valued Lie derivative of the locally Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ with respect to the Filippov set-valued map $F[X] : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$
$\underline{\mathcal{L}}_{\mathcal{F}} f$	Lower set-valued Lie derivative of the lower semicontinuous function $f : \mathbb{R}^d \to \mathbb{R}$ with respect to the set-valued map $\mathcal{F} : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$
$\overline{\mathcal{L}}_{\mathcal{F}} f$	Upper set-valued Lie derivative of the lower semicontinuous function $f : \mathbb{R}^d \to \mathbb{R}$ with respect to the set-valued map $\mathcal{F} : \mathbb{R}^d \to \mathfrak{B}(\mathbb{R}^d)$
$\mathcal{F}$	Set-valued map
sm_Q	Minimum distance function from a point in a convex polygon $Q \subset \mathbb{R}^d$ to the boundary of Q

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